Lecture 14

Recursion

Announcements for Today

Prelim 1

- Tonight at 7:30-9pm
  - A–G (Olin 155)
  - H–K (Olin 165)
  - L–R (Olin 255)
  - S–Z (Upson B17)
- Graded by Noon on Fri
  - Scores will be in CMS
  - In time for drop date
- Make-ups Fri @ 6:30

Other Announcements

- Reading: 5.8 – 5.10
- Assignment 3 now graded
  - **Mean** 93, **Median** 100
  - Typical for this assignment
- Survey for A3 still active
- Assignment 4 posted Fri
  - Uses material from today
  - Due two weeks from today
  - Get started immediately!

Recursion

- **Recursive Definition**:
  A definition that is defined in terms of itself
- **Recursive Function**:
  A function that calls itself (directly or indirectly)

- **Recursion**: If you get the point, stop;
  otherwise, see Recursion
- **Infinite Recursion**: See Infinite Recursion

A Mathematical Example: Factorial

- Non-recursive definition:

  $n! = n \times n\text{-}1 \times \ldots \times 2 \times 1$

  $= n\ (n\text{-}1 \times \ldots \times 2 \times 1)$

- Recursive definition:

  $n! = n\ (n\text{-}1)!$ for $n \geq 0$ **Recursive case**

  $0! = 1$ **Base case**

What happens if there is no base case?

Factorial as a Recursive Function

```
def factorial(n):
    """Returns: factorial of n.
    Pre: n ≥ 0 an int"""
    if n == 0:
        return 1

    return n*factorial(n-1)
```

- $n! = n\,(n-1)!$
- $0! = 1$

Base case(s)

Recursive case

What happens if there is no base case?

Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?

A: $a_8 = 21$
B: $a_8 = 29$
C: $a_8 = 34$
D: None of these.

Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?
- Recursive definition:
  - $a_n = a_{n-1} + a_{n-2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

Why did we need two base cases this time?

Fibonacci as a Recursive Function

```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
            fibonacci(n-2))
```

Base case(s)

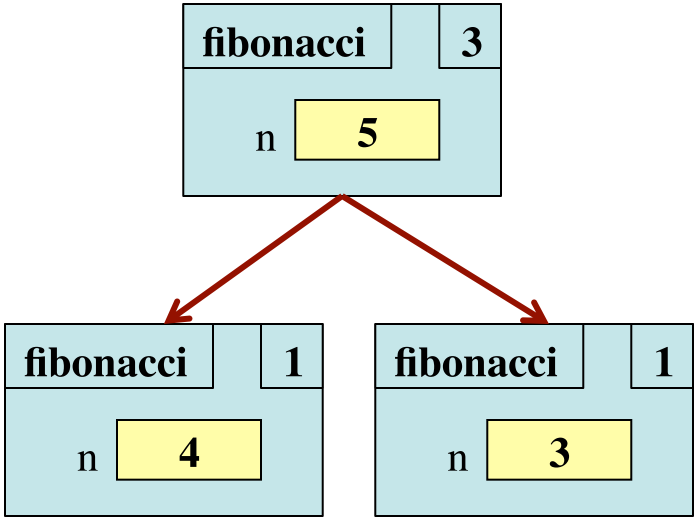
Recursive case

What happens if we forget the base cases?

Fibonacci as a Recursive Function

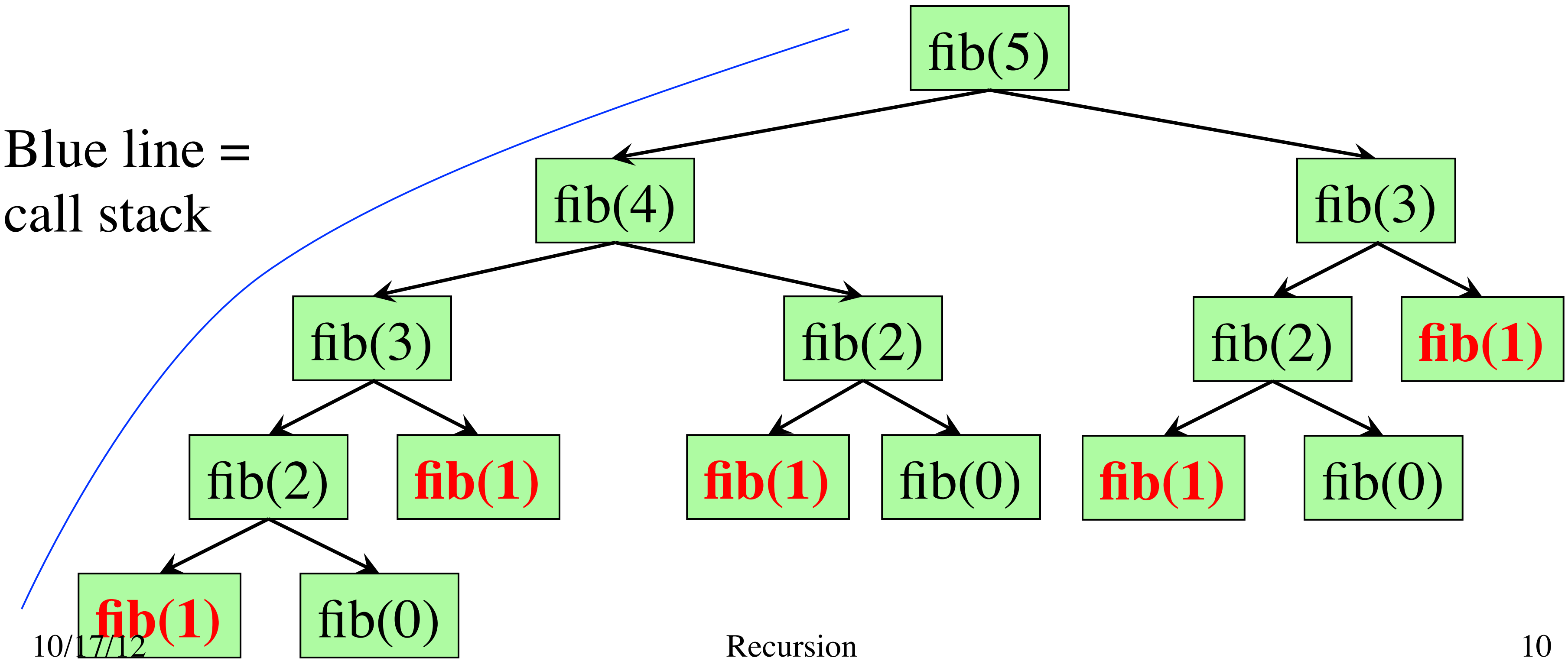
```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
              fibonacci(n-2))
```

- Function that calls itself
  - Each call is new frame
  - Frames require memory
  - ∞ calls = ∞ memory



Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
  - fib(n) has a stack that is always $\leq n$
  - But fib(n) makes a lot of redundant calls



Recursion as a Programming Tool

- Later we will see iteration (loops)
- But recursion is often a good alternative
  - Particularly over sequences (lists, strings)
- Some languages **only** have recursion
  - "Functional languages"; topic of CS 3110

A4: Recursion to draw fractal shapes

String: Two Recursive Examples

```
def length(s):
    """Returns: # chars in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    return 1 + length(s[1:])
```

Imagine len(s) does not exist

```
def num_es(s):
    """Returns: # of ‘e’s in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    return ((1 if s[0] == 'e'
                  else 0) +
               num_es(s[1:]))
```

Two Major Issues with Recursion

- **How are recursive calls executed?**
  - We saw this with the Fibonacci example
  - Use the call frame model of execution
- **How do we understand a recursive function (and how do we create one)?**
  - You cannot trace the program flow to understand what a recursive function does – too complicated
  - You need to rely on the **function specification**

How to Think About Recursive Functions

1. **Have a precise function specification.**
2. **Base case(s):**
   - When the parameter values are as small as possible
   - When the answer is determined with little calculation.
3. **Recursive case(s):**
   - Recursive calls are used.
   - Verify recursive cases with the specification
4. **Termination:**
   - Arguments of calls must somehow get “smaller”
   - Each recursive call must get closer to a base case

Understanding the String Example

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    return ((1 if s[0] == 'e' else 0)
               +  num_es(s[1:]))
```

Base case

Recursive case

	0	1	len(s)
s	H	ello World!	

- Break problem into parts

number of e's in s =
number of e's in s[0]
+ number of e's in s[1:]

- Solve small part directly

number of e's in s =
(1 if s[0] == 'e' else 0)
+ number of e's in s[1:]

Understanding the String Example

- **Step 1:** Have a precise specification

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    # return # of 'e's in s[0]+# of 'e's in s[1:]
    return (1 if s[0] == 'e' else 0) +  num_es(s[1:])
```

Base case

"Write" your return statement using the **specification**

Recursive case

- **Step 2:** Check the base case
  - When s is the empty string, 0 is returned.
  - So the base case is handled correctly.

Understanding the String Example

- **Step 3:** Recursive calls make progress toward termination

```
def num_es(s):
    """Returns: # of ‘e’s in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    # return # of ‘e’s in s[0]+# of ‘e’s in s[1:]
    return (1 if s[0] == 'e' else 0) +  num_es(s[1:])
```

parameter s

argument s[1:] is smaller than parameter s, so there is progress toward reaching base case 0

argument s[1:]

- **Step 4:** Recursive case is correct
  - Just check the specification

Exercise: Remove Blanks from a String

1. Have a precise specification

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

2. Base Case: the smallest String s is ''.

```
if s == '':
    return s
```

3. Other Cases: String s has at least 1 character.

```
return (s[0] with blanks removed) + (s[1:] with blanks removed)
```

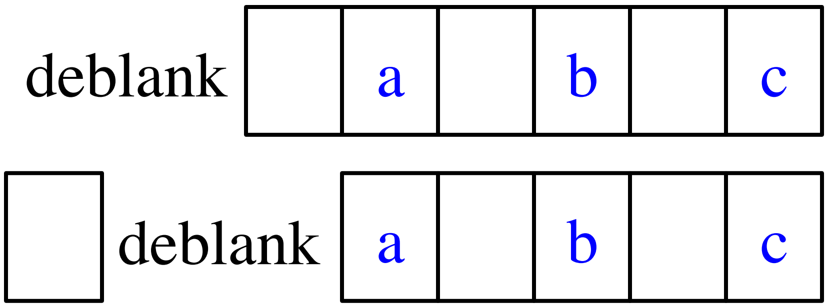
```
('' if s[0] == ' ' else s[0])
```

What the Recursion Does

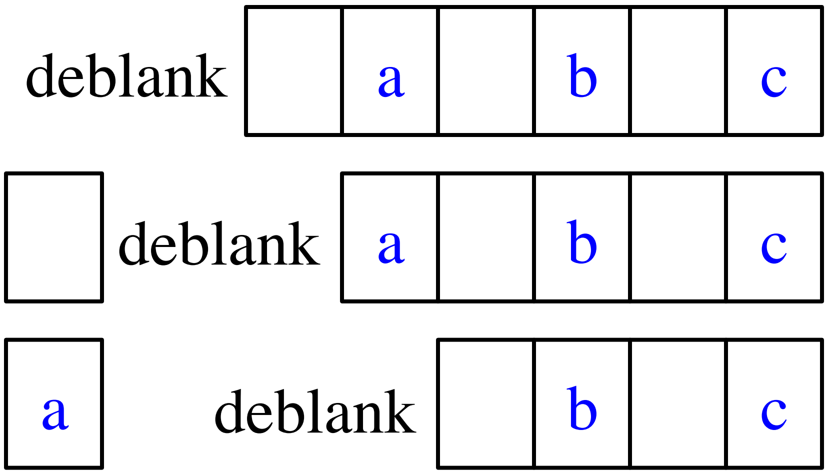
deblank

	a		b		c

What the Recursion Does



What the Recursion Does



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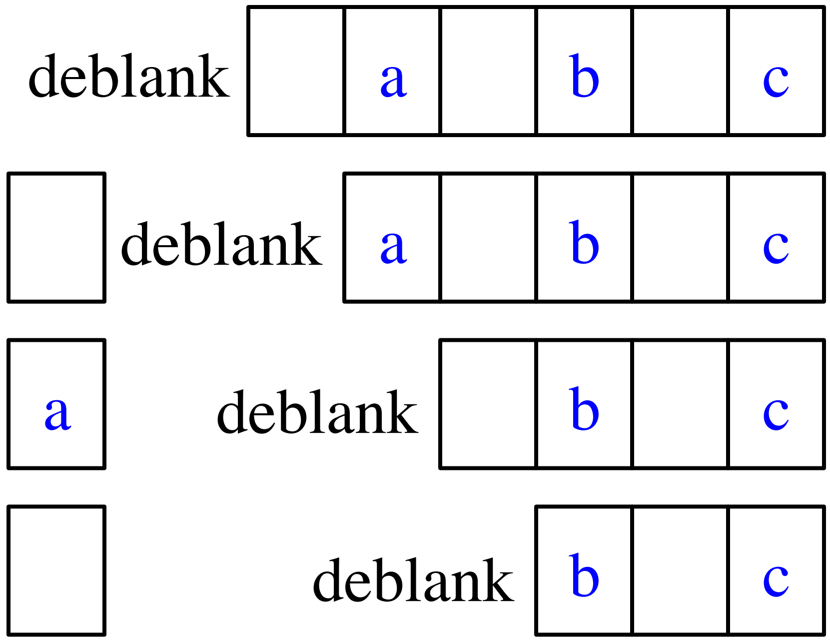

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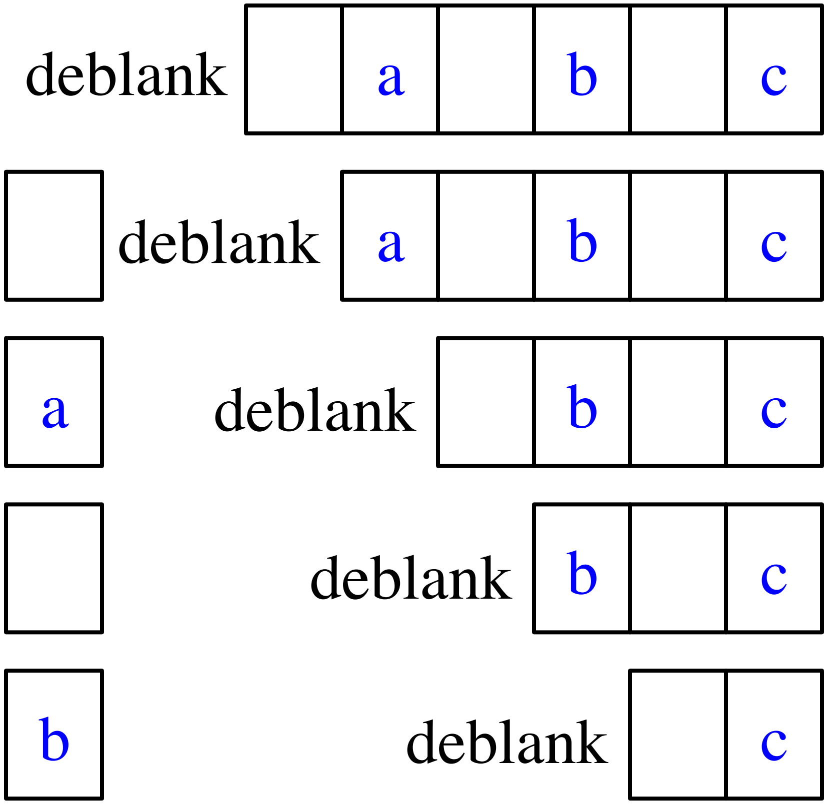

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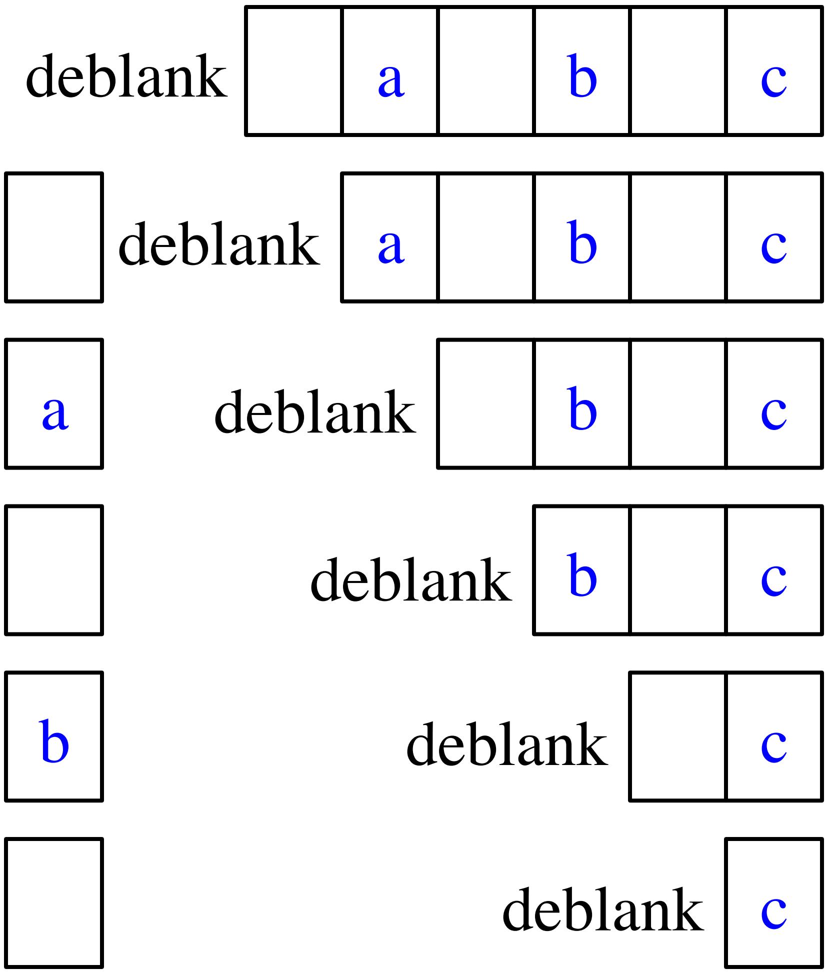

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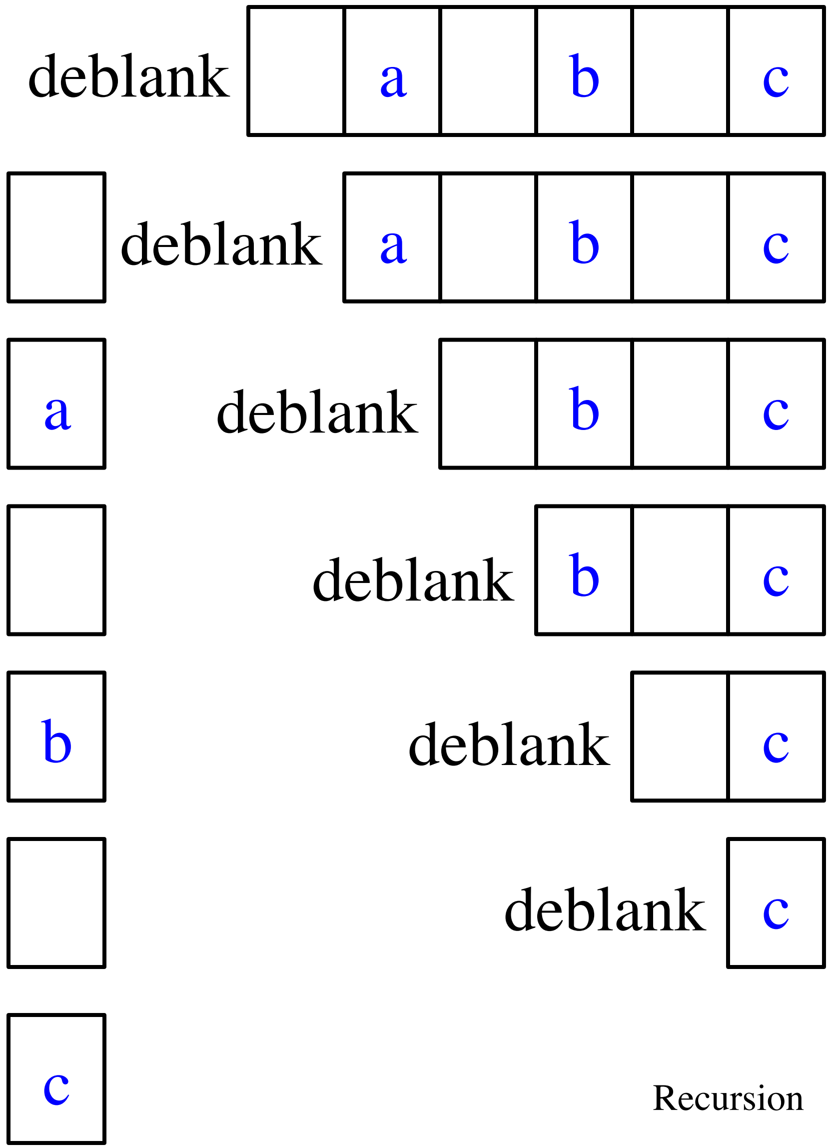

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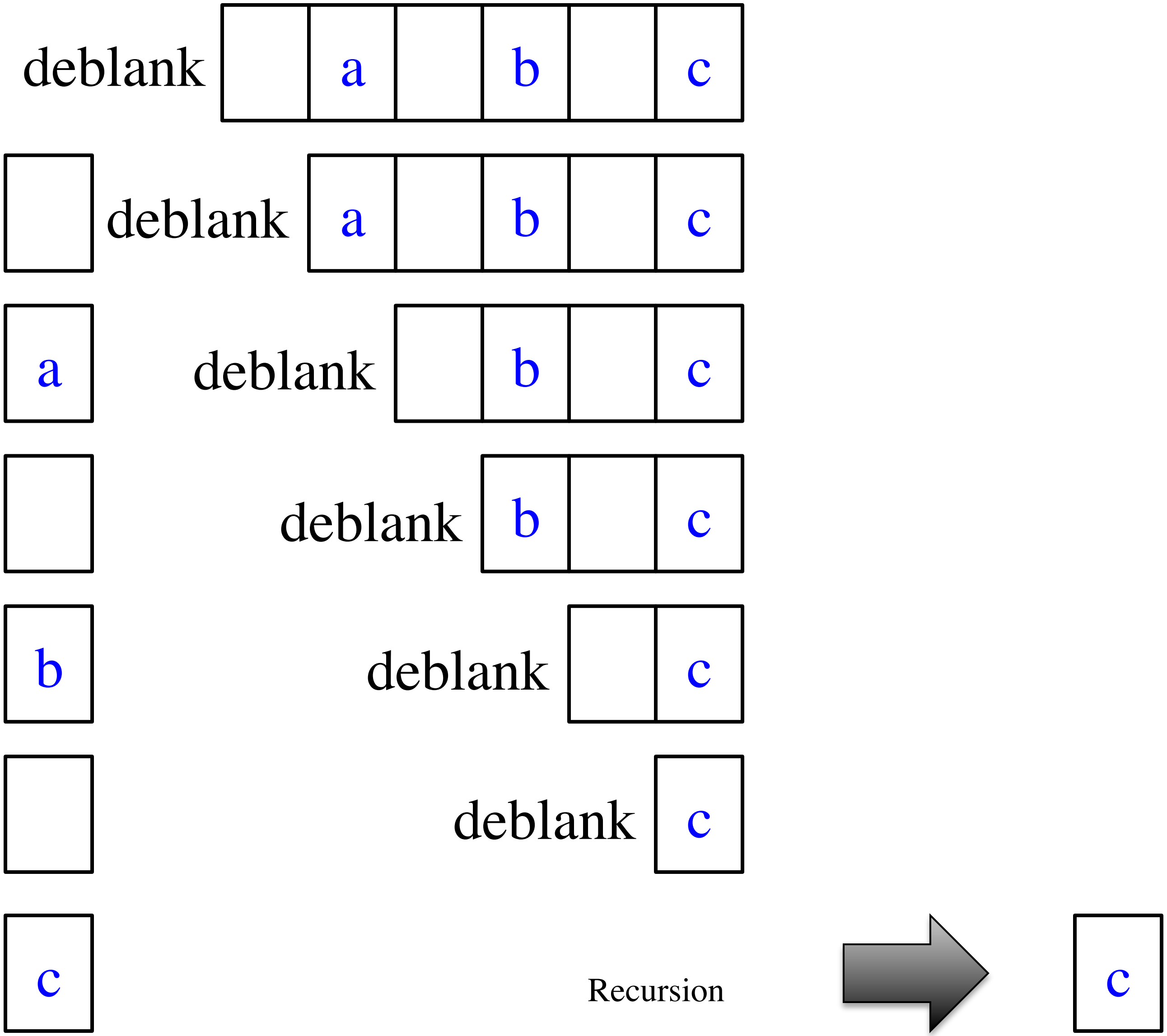

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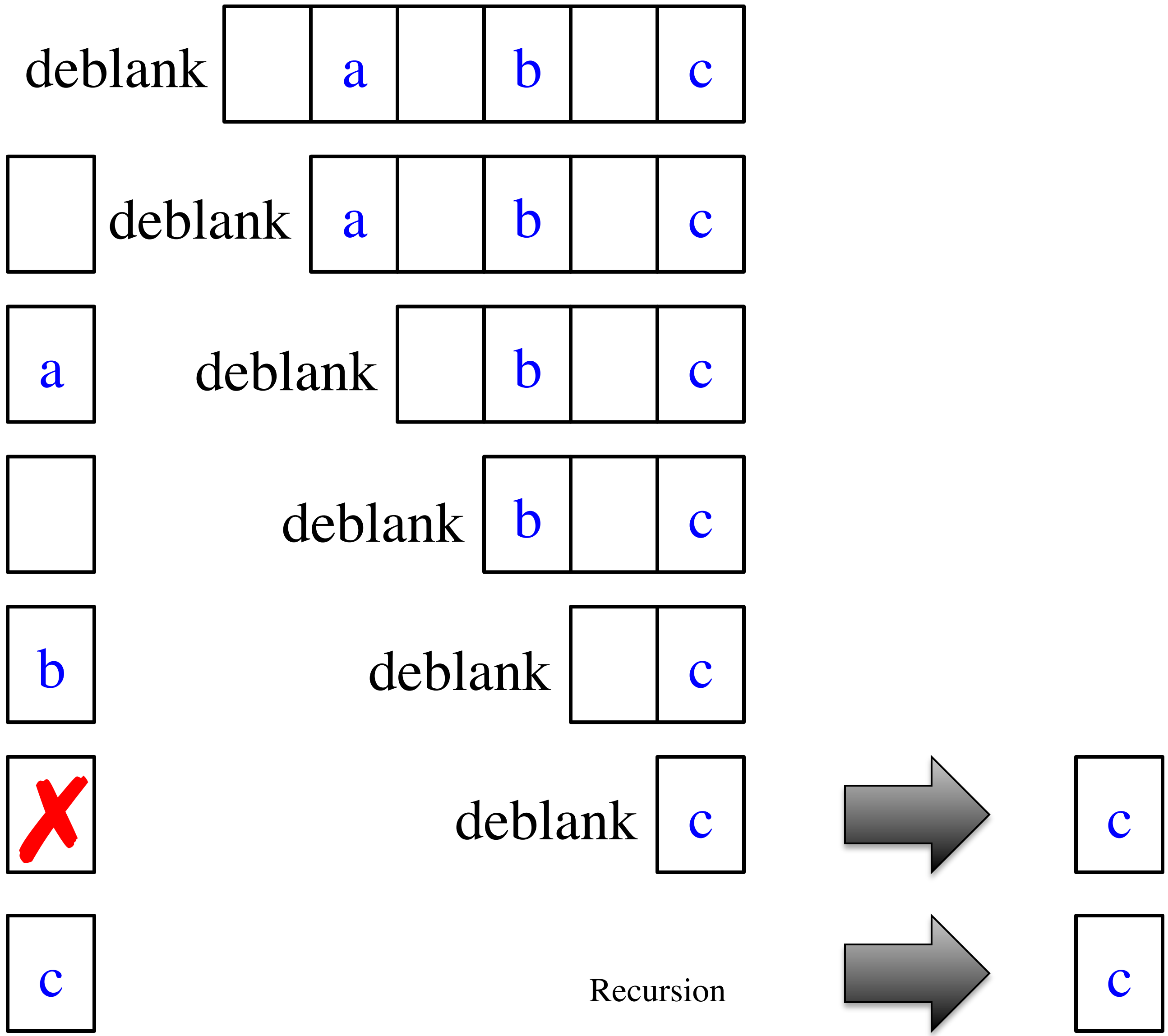

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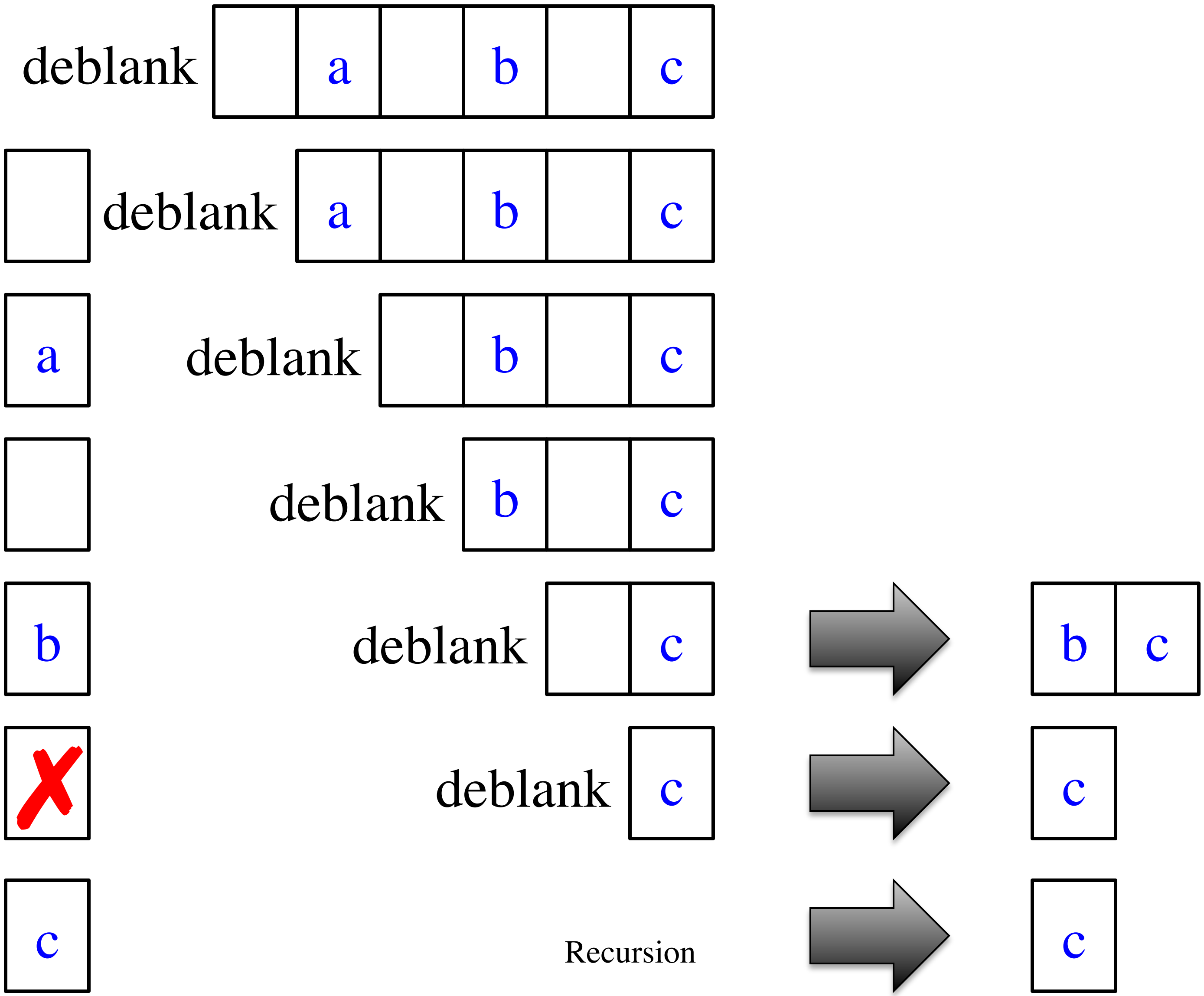

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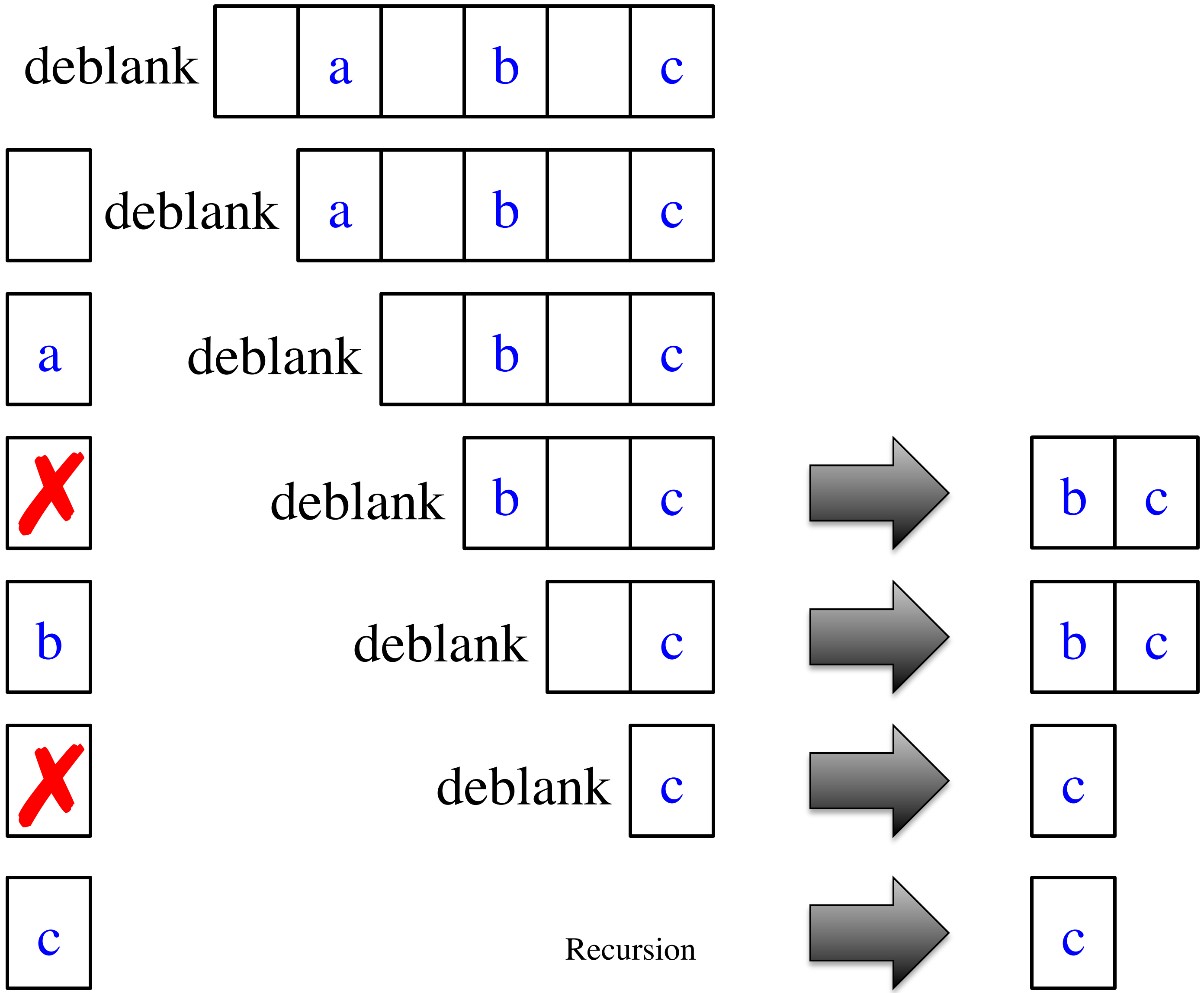

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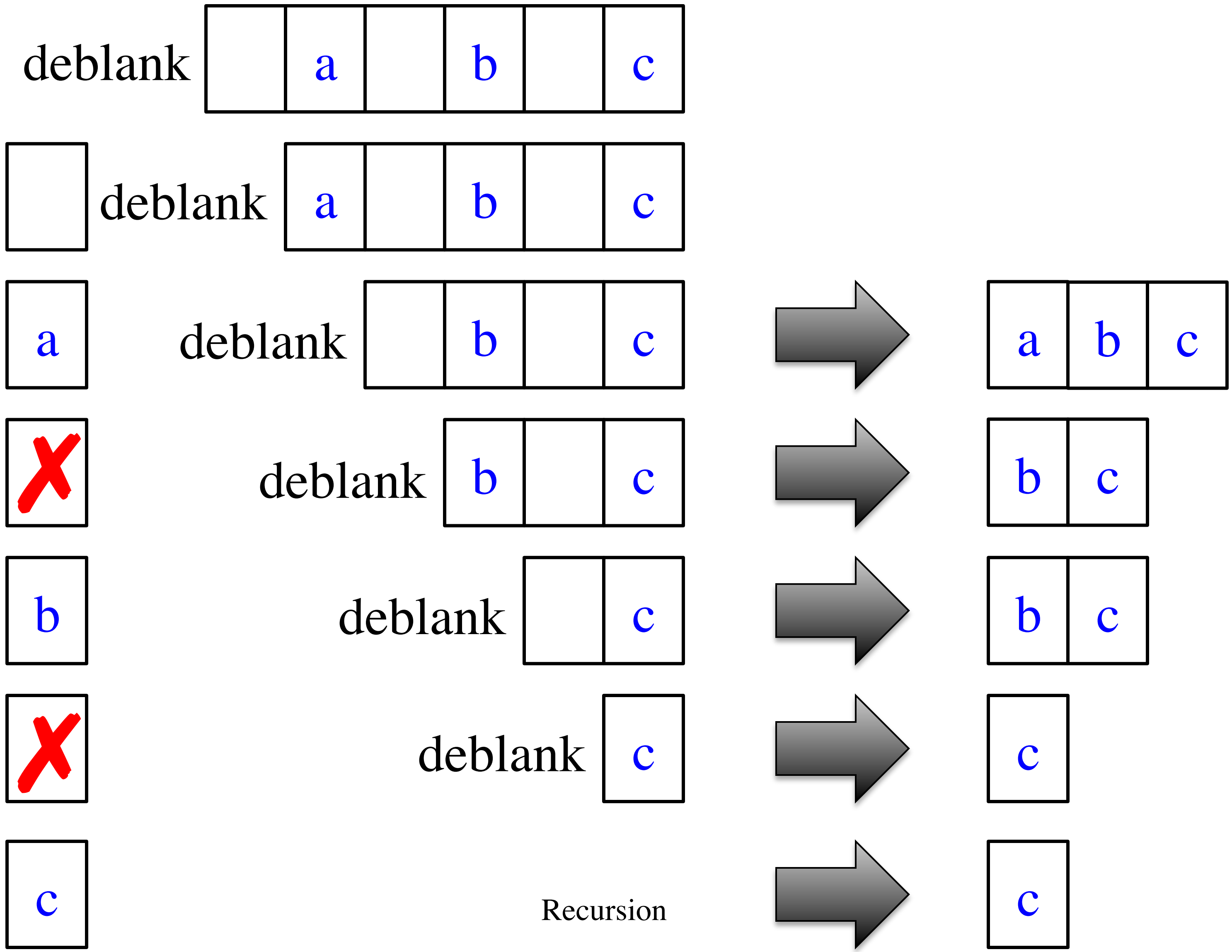

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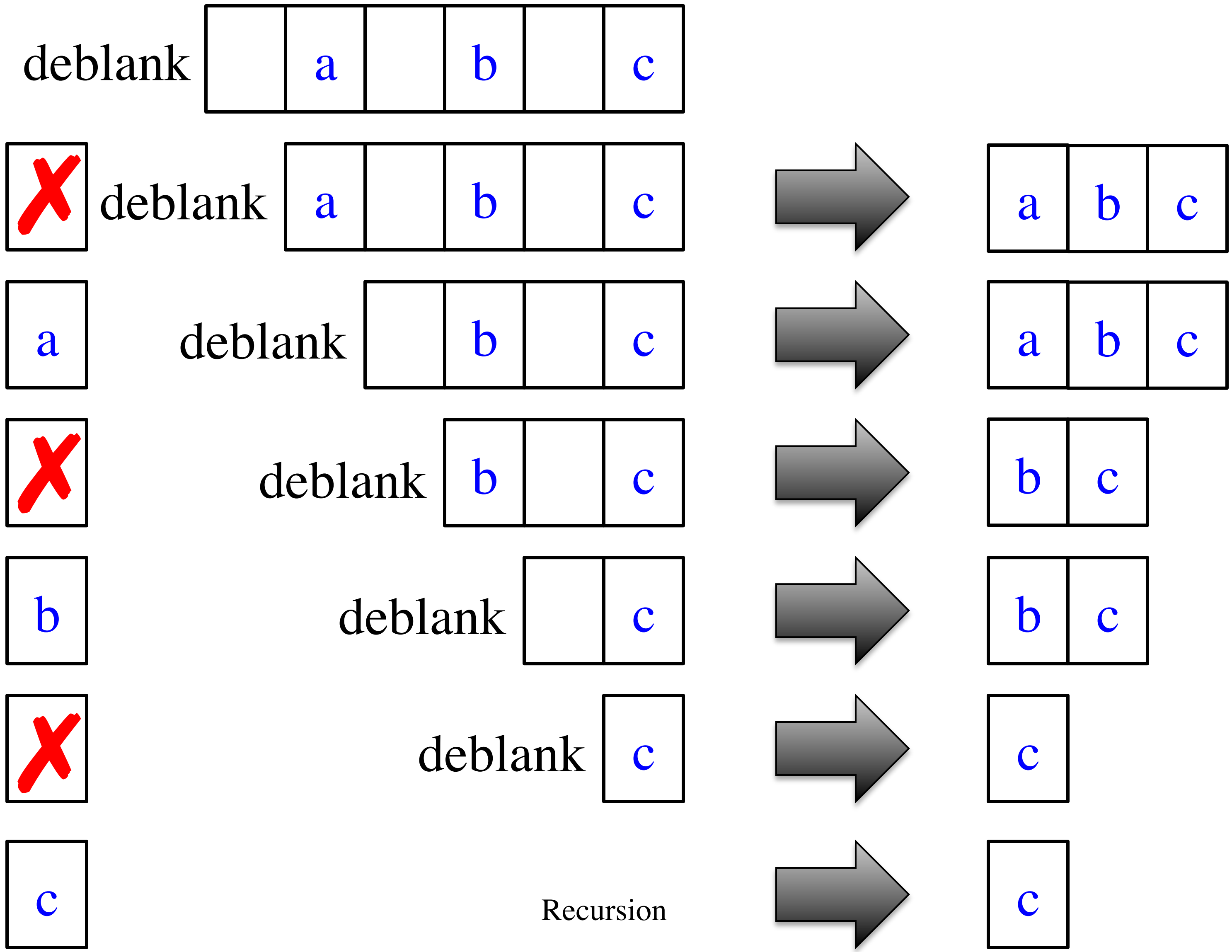

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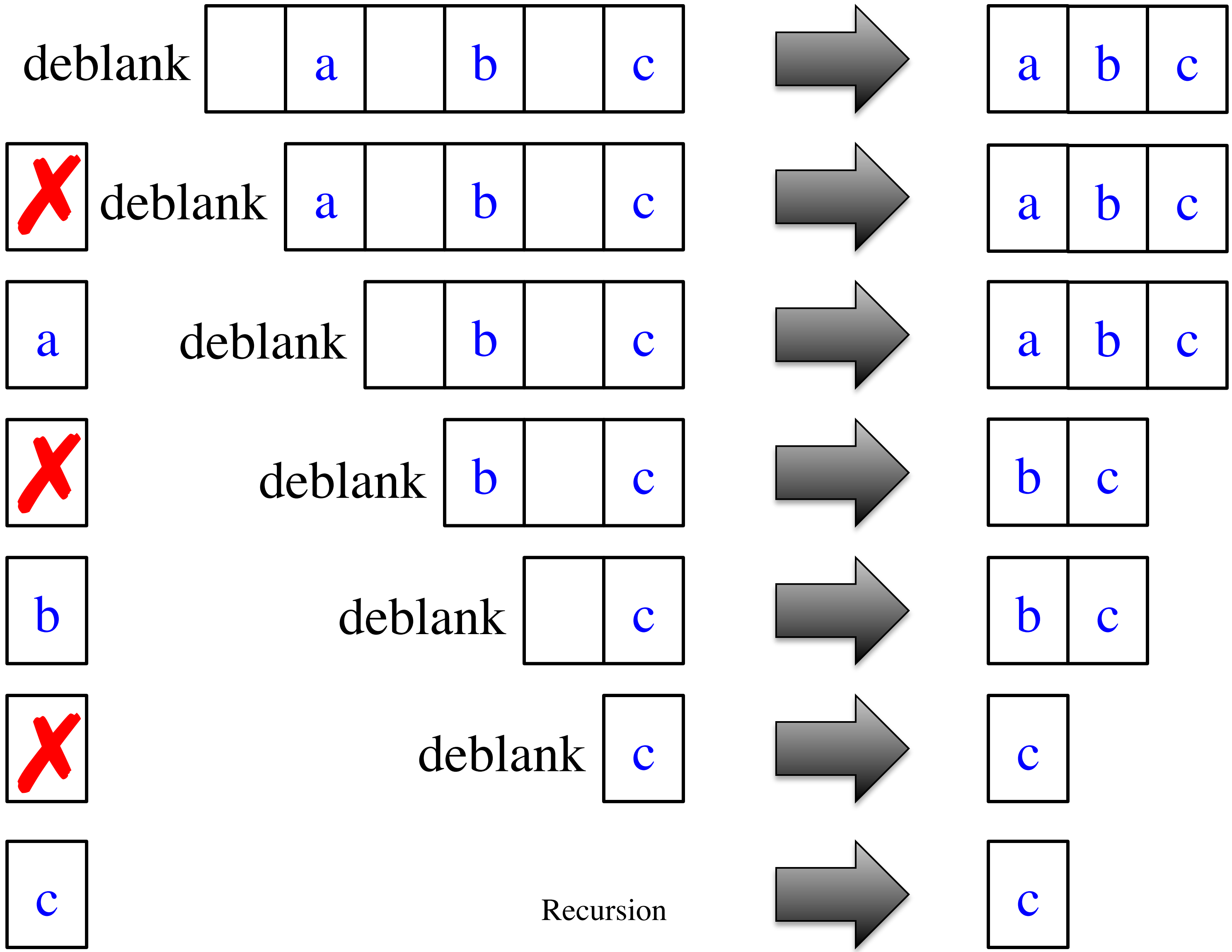

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s with blanks removed"""
    if s == '':
        return s

    # s is not empty
    if s[0] is a blank:
        return s[1:] with blanks removed

    # s not empty and s[0] not blank
    return (s[0] +
            s[1:] with blanks removed)
```

- Sometimes easier to break up the recursive case
  - Particularly om small part
  - Write recursive case as a sequence of if-statements
- Write code in *pseudocode*
  - Mixture of English and code
  - Similar to top-down design
- Stuff in **red** looks like the function specification!
  - But on a smaller string
  - Replace with `deblank(s[1:])`

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s with blanks removed"""
    if s == '':
        return s

    # s is not empty
    if s[0] in string.whitespace:
        return deblank(s[1:])

    # s not empty and s[0] not blank
    return (s[0] +
            deblank(s[1:]))
```

- Check the four points:
  1. Precise specification?
  2. Base case: correct?
  3. Recursive case: progress toward termination?
  4. Recursive case: correct?

Expression: `x in thelist` returns `True` if `x` is a member of list `thelist` (and `False` if it is not)

Next Time: A Lot of Examples