Recursion

- **Recursive Definition**:
  A definition that is defined in terms of itself
- **Recursive Function**:
  A function that calls itself (directly or indirectly)

- **Recursion**: If you get the point, stop;
  otherwise, see Recursion
- **Infinite Recursion**: See Infinite Recursion

A Mathematical Example: Factorial

- Non-recursive definition:
  $n! = n \times n\text{-}1 \times \ldots \times 2 \times 1$
  $= n\,(n\text{-}1 \times \ldots \times 2 \times 1)$
- Recursive definition:
  $n! = n\,(n\text{-}1)!$ for $n \geq 0$ **Recursive case**
  $0! = 1$ **Base case**

What happens if there is no base case?

Example: Fibonnaci Sequence

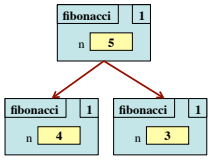
- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...
  $a_0\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6$
  - Get the next number by adding previous two
  - What is a_8?
- Recursive definition:
  - $a_n = a_{n-1} + a_{n-2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

Why did we need two base cases this time?

Fibonacci as a Recursive Function

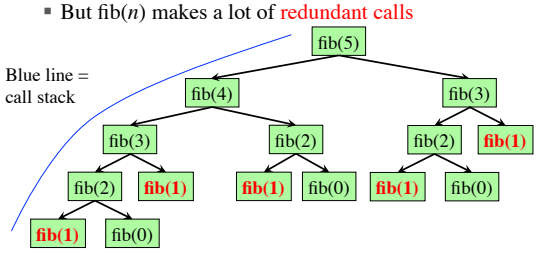
```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
            fibonacci(n-2))
```

- Function that calls itself
  - Each call is new frame
  - Frames require memory
  - ∞ calls = ∞ memory



Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
  - fib(n) has a stack that is always $\leq n$
  - But fib(n) makes a lot of redundant calls



String: Two Recursive Examples

```
def length(s):
    """Returns: # chars in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    return 1 + length(s[1:])
```

Imagine len(s) does not exist

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    return ((1 if s[0] == 'e'
             else 0) +
            num_es(s[1:]))
```

How to Think About Recursive Functions

1. **Have a precise function specification.**
2. **Base case(s):**
   - When the parameter values are as small as possible
   - When the answer is determined with little calculation.
3. **Recursive case(s):**
   - Recursive calls are used.
   - Verify recursive cases with the specification
4. **Termination:**
   - Arguments of calls must somehow get "smaller"
   - Each recursive call must get closer to a base case

Understanding the String Example

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # {s is empty}
    if s == '':
        return 0                          # Base case

    # { s at least one char }
    return ((1 if s[0] == 'e' else 0)
            + num_es(s[1:]))              # Recursive case
```

	0	1		len(s)
s	H	ello World!		

- Break problem into parts

  number of e's in s =
  number of e's in s[0]
  + number of e's in s[1:]

- Solve small part directly

  number of e's in s =
  (1 if s[0] = 'e' else 0)
  + number of e's in s[1:]

Understanding the String Example

- **Step 1:** Have a precise specification

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # {s is empty}
    if s == '':
        return 0                          # Base case

    # { s at least one char }
    # return # of 'e's in s[0]+# of 'e's in s[1:]
    return (1 if s[0] == 'e' else 0) + num_es(s[1:])   # Recursive case
```

"Write" your return statement using the **specification**

- **Step 2:** Check the base case
  - When s is the empty string, 0 is returned.
  - So the base case is handled correctly.

Understanding the String Example

- **Step 3:** Recursive calls make progress toward termination

```
def num_es(s):                            # parameter s
    """Returns: # of 'e's in s"""
    # {s is empty}
    if s == '':
        return 0

    # { s at least one char }
    # return # of 'e's in s[0]+# of 'e's in s[1:]
    return (1 if s[0] == 'e' else 0) + num_es(s[1:])   # argument s[1:]
```

argument s[1:] is smaller than parameter s, so there is progress toward reaching base case 0

- **Step 4:** Recursive case is correct
  - Just check the specification

Exercise: Remove Blanks from a String

1. Have a precise specification

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

2. Base Case: the smallest String s is ''.

```
if s == '':
    return s
```

3. Other Cases: String s has at least 1 character.

```
return (s[0] with blanks removed) + (s[1:] with blanks removed)
```

('' if s[0] == ' ' else s[0])

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s with blanks removed"""
    if s == '':
        return s

    # s is not empty
    if s[0] is a blank:
        return s[1:] with blanks removed

    # s not empty and s[0] not blank
    return (s[0] +
            s[1:] with blanks removed)
```

- Sometimes easier to break up the recursive case
  - Particularly om small part
  - Write recursive case as a sequence of if-statements
- Write code in *pseudocode*
  - Mixture of English and code
  - Similar to top-down design
- Stuff in **red** looks like the function specification!
  - But on a smaller string
  - Replace with deblank(s[1:])