Math Olympiad 4 roundup

1. Each letter in the ordered list A, B, C, D, E, . . . , Y, Z represents a number, not necessarily different from each other, so two letters may represent the same number. The sum of any 4 consecutive numbers is 30. If A=3, C=9, and D=10, what is the value of J?

   Answer:

   8

2. In the previous problem, what is the value of Z?

   Answer:

   Also 8!

3. Each letter in the ordered list A, B, C, D, E again represents a number. The sum of any three consecutive numbers is 16. If A=1, what is D?

   Answer:

   Must be the same as A: 1

4. The clock at Jacobi Towers is also slow. Every day at noon, Theo corrects it from 11:37 to 12:00pm. He misses correcting it for 5 days. What time does it read when he shows up at noon to correct it on the 6th day?

   Answer:

   6 × 23 = 148. So it reads 9:42am.

5. Two watches are set correctly at 7:00 AM. One watch gains 3 minutes every two hours. The other watch loses 1 minute every two hours. At what time the next day will the faster watch be exactly one hour ahead of the slower watch? (indicate AM or PM)

   Answer:

   The faster watch gains by 2 minutes per hour, so it will take 15 hours: 1PM

6. Theo lost his adjustment wrench. How many days will it take before the Jacobi clock towers read the correct time again at noon?

   Answer:
   After n days the time reads $23 \times n$ minutes slow. For this to be 12:00am or 12:00pm, the number of minutes slow must be a multiple of the number of minutes in a half a day, which is 12×60 = 720. The least common multiple of 720 and 23 is 23×720, since they have no common prime factors. Therefore it will take 720 days, or almost two years!

7. How many numbers between 1 and 1000 are divisible by 5 and by 7?

   Answer:
   If a number is a multiple of both 5 and 7, it must be a multiple of the least common multiple of 5 and 7, which is 35. The first multiple of 35 is 35. Dividing 1000 by 35, we get 28 with a remainder, so the multiples of 35 go from 1×35 to 28×35. So there are $\boxed{28}$.

8. How many numbers between 1 and 1000 are divisible by 10 but not by 15?

   Answer:
   The least common multiple of 10 and 15 is 2×3×5 = 30. There are 1000/10 = 100 numbers divisible by 10. Since 1000/30 = 33 (with a remainder), there are 33 numbers divisible by 30. Therefore there are $100 - 33 = \boxed{67}$ *numbers divisible by 10 but not 15.*

9. How many numbers between 1 and 1000 are divisible by 6 or by 8, but not by 9?

Answer:

This is trickier! The following Venn diagram may help us think about it. There are $\lfloor 1000/6 \rfloor = 166$ *numbers divisible by 6 and* $1000/8 = 125$ *divisible by 8. But we don't want to double-count the numbers divisible by both. Since LCM(6,8) is 24, we need to subtract the* $\lfloor 1000/24 \rfloor = 41$ *numbers divisible by both. So that gives us* $166 + 125 - 41 = 250$ *numbers divisible by either 6 or 8. Now we need to remove the ones divisible by 9. Since LCM(9,24) = 3×3×2×2×2 = 72 = LCM(8,9), all the numbers in the intersection of the blue and green circles are actually in the center. So we just need to remove the numbers divisible by both 6 and 9. These are divisible by 18, so there are 55 of them.* $250 - 55 = \boxed{195}$.