On NP-completeness in Linear Logic

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Introduction

Linear Logic has been introduced by Girard [Girard,1987]. In the current paper we consider multiplicative fragments of Linear Logic (i.e. fragments with two connectives $\otimes$ and $\multimap$) containing only two constants: 1 and $\perp$ ($MLL(\perp, 1)$). The inference rules of the multiplicative fragment of Linear Logic are given in Table 1.

	$\frac{\Pi,A,B,\Gamma\vdash\Delta}{\Pi,B,A,\Gamma\vdash\Delta}$		$\frac{\Gamma\vdash\Delta,A,B,\Pi}{\Gamma\vdash\Delta,B,A,\Pi}$
I	$\frac{}{A\vdash A}$		
$L\otimes$	$\frac{A,B,\Gamma\vdash\Delta}{A\otimes B,\Gamma\vdash\Delta}$	$R\otimes$	$\frac{\Gamma\vdash A,\Delta \quad \Pi\vdash B,\Sigma}{\Gamma,\Pi\vdash A\otimes B,\Delta,\Sigma}$
$L\multimap$	$\frac{\Gamma\vdash A,\Delta \quad B,\Pi\vdash\Sigma}{A\multimap B,\Gamma,\Pi\vdash\Delta,\Sigma}$	$R\multimap$	$\frac{A,\Gamma\vdash B,\Delta}{\Gamma\vdash A\multimap B,\Delta}$
$L\perp$	$\frac{}{\perp\vdash}$	$R\perp$	$\frac{\Gamma\vdash\Delta}{\Gamma\vdash\perp,\Delta}$
$L1$	$\frac{\Gamma\vdash\Delta}{1,\Gamma\vdash\Delta}$	$R1$	$\frac{}{\vdash 1}$
CUT	$\frac{\Gamma\vdash A,\Delta \quad A,\Pi\vdash\Sigma}{\Gamma,\Pi\vdash\Delta,\Sigma}$		

Table 1.
The Inference Rules of Multiplicative Fragment of Linear Logic.

Remark. The Cut rule can be eliminated ([Girard,1987]).

We shall also consider the multiplicative fragment of Linear Logic with Weakening rule MLLW (see Table 2).

$\frac{\Gamma\vdash\Delta}{A,\Gamma\vdash\Delta}$	$\frac{\Gamma\vdash\Delta}{\Gamma\vdash A,\Delta}$

Table 2.
The Weakening Rule.

In [Lincoln *et al.* 1990] it was demonstrated that the MLLW is NP-complete, and in [Kanovich,1991] it was proved that the MLL is NP-complete too. In order to find the exact bound of NP-completeness in Linear Logic, the Horn fragment of Linear Logic (HLL) has been introduced in [Kanovich,1992], i.e. the fragments containing only sequents of the form

$$\Gamma \vdash \Delta,$$

where Δ contains only simple products, and Γ contains simple products and implications of the form $(X \multimap Y)$, where X and Y are simple products. (A simple product is a tensor product of a number of positive literals and constants). It is clear that HLL is a proper fragment of MLL.

Below we shall use the following notational conventions:

$MLL_{[k]}(\perp, 1)$ and $HLL_{[k]}(\perp, 1)$—are respectively multiplicative and Horn fragments with k literals and constants $\perp$ and 1;

$MLLW(\perp, 1)$ and $HLLW(\perp, 1)$—are multiplicative and Horn fragments with Weakening rule containing only constants $\perp$ and 1;

$HLL()$ and $MLL()$—are the fragments with no constants;

and so on.

Definition 1

For any literal p, the number $\#_p A$ is defined as follows:
$\#_p p = 1$;
$\#_p q = 0$, if q is a literal different from p;
$\#_p (A \otimes B) = \#_p A + \#_p B$;
$\#_p (A \multimap B) = \#_p B - \#_p A$;
$\#_p \Gamma = \sum\limits_{A \in \Gamma} \#_p A$.

In this paper we shall need the conception of equipartition problem, which has been introduced in [Garey and Jonson,1979].

The equipartition problem is defined as follows:

Given k-dimensional vector of natural numbers

$$s = (s_1, s_2, \ldots, s_k),$$

such as $b/4 < s_i < b/2$ for every $i = 1, \ldots, k$, where $k = 3m$, and $b = \frac{1}{m} \sum s_i$.

Find whether the following assertion is correct:

(1) the set $\{1, 2, \ldots, k\}$ can be partitioned into m disjoint sets $S_1, S_2, \ldots, S_m$ (three elements in each) such that $\sum\limits_{i \in S_j} s_i = b$, for $j = 1, \ldots, m$.

In [Garey and Jonson,1979] it was demonstrated that the equipartition problem is NP-complete.

Summary

In [Kanovich,1992] it was shown that both HLL's with Weakening and without Weakening are NP-complete, even if they contain occurrences of only two literals and do not contain constants at all. On the other hand, it was observed in [Kanovich,1992] that the one-literal HLL's (both with Weakening and without Weakening) are decidable in linear time if they contain no constants.

In [Lincoln and Winkler,1992] it was proved, that $MLL_{[0]}(\bot)$ is NP-complete. And Kanovich demonstrated NP-completness of $MLL_{[1]}()$ and $MLLW_{[1]}$.

In the current paper we shall give our proofs of the three last facts (NP-completness of $MLL_{[0]}(\bot)$, $MLL_{[1]}()$ and $MLLW_{[1]}$). Moreover, we shall show that these fragments are NP-complete, even if they do not contain any nested implication.

We shall prove also the following assertions:

- The Horn fragment of Linear Logic with one literal and one constant $\bot$ is NP-complete;
- But the similar fragment with Weakening turns out to be decidable in linear time;
- The Horn fragment of Linear Logic with constant $\bot$ and without literals turns out to be decidable in linear time.
- And the multiplicative fragment of Linear Logic with Weakening containing only one constant $\bot$ and do not any literals is decidable in linear time;

Together these assertions provide exact bound of NP-completeness for the multiplicative fragments of Linear Logic (both with Weakening and without Weakening). These results are represented in Table 3.

k	$HLL_{[k]}()$	$HLL_{[k]}(\bot)$	$MLL_{[k]}()$	$MLL_{[k]}(\bot)$
0	—	$O(n)$	—	NP
1	$O(n)$	NP	NP	NP
≥ 2	NP	NP	NP	NP

k	$HLL_{[k]}W()$	$HLL_{[k]}W(\bot)$	$MLL_{[k]}W()$	$MLL_{[k]}W(\bot)$
0	—	$O(n)$	—	$O(n)$
1	$O(n)$	$O(n)$	NP	NP
≥ 2	NP	NP	NP	NP

Table 3. Exact bound of NP-completeness
for the multiplicative fragments of Linear Logic
with Weakening and without Weakening.

Remark. The constant 1 does not influence the complexity level of all these fragments.

1 NP-completeness

1.1 Fragments of MLL without Constants

In [Kanovich,1992] it was shown that HLL is NP-complete, even if it contains occurrences of only two literals and do not contain constants at all. In order to prove it, for every vector s from equipartition problem the following sequent was considered:

$$q^b, \Gamma \vdash q^b, \tag{2}$$

where Γ consists of the n formulas:

$$q^{s_i} \multimap r^{s_i}, \quad (\text{for } i = 1, \dots, n)$$

and k copies of formulas:

$$r^b \multimap q^b.$$

Lemma 1. [Kanovich,1992]
Let s and b be as in definition of equipartition problem above, then sequent (2) is derivable in Linear Logic if and only if the assertion (1) is correct for vector s.

Theorem 1. [Kanovich,1992]
$HLL_{[2]}()$ is NP-hard.

Similarly, we shall demonstrate that the multiplicative fragment of Linear Logic (both with Weakening and without Weakening) containing occurrences of only one literal and containing no constants is NP-complete. (These facts were already proved by Kanovich, but we shall give here another proof.)

For every vector s and numbers m and b (see equipartition problem) we consider the sequent

$$\Gamma \vdash (q^M \multimap q^{M+b})^m, \tag{3}$$

where Γ consist of k formulas

$$q^M \multimap q^{M+s_i}\ , \quad (\text{for } i = 1, \dots, k,)$$

and M is any number greater than $\sum s_i$.

Lemma 2.
Let vector s and numbers b and m be as in definition of equipartition problem above. If $\sum s_i = mb$, then the following three conditions are equivalent:

(i) The sequent (3) is derivable in Linear Logic.

(ii) The sequent (3) is derivable in Linear Logic with the Weakening rule.

(iii) Assertion (1) is correct.

Lemma 2.1.
If the sequent $\Gamma \vdash C$ is derivable in $HLLW()$ and

$$\forall A \in \Gamma \quad \#_p A > 0,$$

then $\#_p\Gamma \geq \#_p C$.

Proof.

This is easy to prove by induction on derivation of the sequent $\Gamma \vdash C$. ■

Proof of Lemma 2.

(i) $\Longrightarrow$(ii) This implication is evident.

(iii)$\Longrightarrow$ (i) Let the assertion (1) be correct. Hence, for any $j = 1, \ldots, m$, if $S_j = \{i_1, i_2, i_3\}$, then the sequent

$$q^M \multimap q^{M+s_{i_1}}\ , q^M \multimap q^{M+s_{i_2}}\ , q^M \multimap q^{M+s_{i_3}}\ \vdash q^M \multimap q^{M+b}$$

is derivable in LL. And the sequent (3) is derived from these m sequents by the $(R\otimes)$ rule.

(ii) $\Longrightarrow$ (iii) We shall prove this implication by induction on m. If $m = 1$, then it is evident. Let $m > 1$. The sequent (3) can be derived by one of the rules $(R\otimes)$, $(L\multimap)$ or Weakening.

Case 1. The sequent is derived by rule $(R\otimes)$.

In this case the derivation was the following:

$$\frac{\Gamma_1 \vdash (q^M \multimap q^{M+b})^{m_1} \qquad \Gamma_2 \vdash (q^M \multimap q^{M+b})^{m_2}}{\Gamma_1, \Gamma_2 \vdash (q^M \multimap q^{M+b})^{m_1+m_2}}\,(R\otimes).$$

Then the set $\{1, \ldots, k\}$ can be segmented into two disjoint sets I_1 and I_2, such that for $l = 1, 2$ the multiset Γ_l consists of the formulas

$$q^M \multimap q^{M+s_i}\ ,$$

where i runs through the set I_l. According to Lemma 2.1

$$\sum_{i \in I_l} s_i = \#_q\Gamma_l \geq \#_q(q^M \multimap q^{M+b})^{m_l} = m_l b \qquad (\text{for } l = 1, 2). \tag{4}$$

Since

$$\sum_{i=1}^{n} s_i = mb = (m_1 + m_2)b,$$

inequalities (4) turn into equality. Therefore, according to the induction hypothesis the sets I_l $(l = 1, 2)$ can be segmented into disjoint sets $S_{l,1}, S_{l,2}, \ldots, S_{l,m_l}$ (three elements in each) such that

$$\sum_{i \in S_{l,j}} s_i = b \qquad \text{for } j = 1, \ldots, m.$$

So, the set $\{1, \ldots, k\}$ can be segmented into m disjoint sets $S_{1,1}, \ldots, S_{1,m_1}$, $S_{2,1}, \ldots, S_{2,m_2}$ such that for any of these sets S

$$\sum_{i \in S} s_i = b.$$

Case 2. The sequent is derived by rule $(L\multimap)$.

In this case the derivation was the following:

$$\frac{\Gamma_1, q^{M+s_i} \vdash (q^M \multimap q^{M+b})^m \qquad \Gamma_2 \vdash q^M}{\Gamma \vdash (q^M \multimap q^{M+b})^m}\,(L\multimap).$$

And we have got a contradiction with Lemma 2.1, because

$$\#_q\Gamma_2 \leq \#_q\Gamma = \sum s_i < M$$

<u>Case 3.</u> The sequent is derived by Weakening. In this case we again have got a contradiction with Lemma 2.1. ■

Theorem 2.
Both $MLL_{[1]}()$ and $MLLW_{[1]}()$ are NP-hard.
Proof.
This follows from NP-completeness of equipartition problem [Garey and Jonson,1979] and Lemma 2. ■

Corollary.
Both $MLL_{[1]}()$ and $MLLW_{[1]}()$ are NP-complete.
Proof.
This follows from Theorem 2 and NP-completeness of MLL [Kanovich,1991] and $MLLW$ [Lincoln *et al.* 1990]. ■
Remark. Let us note that in this proof we use only formulas without nested implications. Hence, both $MLL_{[1]}()$ and $MLLW_{[1]}()$ are NP-complete even if they contain no nested implications, although the Horn fragments are decidable in linear time.

1.2 Fragments of MLL with Constant $\perp$

We shall prove that both $HLL_{[1]}(\perp)$ and $MLL_{[0]}(\perp)$ are NP-complete.
Definition 1.
A formula is "good" if it does not contain subformula of the form: $A \multimap \perp$; and a sequent is "good" if it contains only "good" formulas.
Lemma 3.
If a "good" sequent $\Gamma \vdash C$ is derivable in $MLL(\perp)$ then this sequent is derivable without rules $(L\perp)$ and $(R\perp)$ (i.e a "good" sequent $\Gamma \vdash C$ is derivable in $MLL(\perp)$ if and only if a sequent $\Gamma' \vdash C'$ is derivable in $MLL()$, where $'$ stands for substitution of a new literal f for constant $\perp$).
To prove Lemma 3 we shall prove Lemma 3.1.
Lemma 3.1.
If a "good" sequent $\Gamma \vdash$ is derivable in $MLL(\perp)$, then this sequent is an axiom, i.e. $\Gamma = \perp$.
Proof.
Note that a cut free derivation of a "good" sequent contains only "good" sequents.
We prove lemma by induction on the sequent derivation length.

There are two cases to be considered: sequent $\Gamma \vdash$ is derived by the rule $(L\otimes)$ or by the rule $(L\multimap)$.

<u>Case 1.</u> The sequent is derived by the rule $(L\otimes)$.

Then the derivation was the following:

$$\frac{A, B, \Pi \vdash}{A \otimes B, \Pi \vdash}\,(L\otimes).$$

In this case the sequent $A, B, \Pi \vdash$ would be "good", and we have got a contradiction with the inductive hypothesis.

<u>Case 2.</u> The sequent is derived by the rule $(L\multimap)$.

In this case the derivation was the following:

$$\frac{\Pi \vdash A \qquad B, \Delta \vdash}{A \multimap B, \Pi, \Delta \vdash}\,(L\multimap).$$

According to the inductive hypothesis, $B = \bot$, $\Delta = \Lambda$. But then the sequent

$$A \multimap B, \Pi, \Delta \vdash$$

is not "good". We have got a contradiction. ■

Proof of Lemma 3.

The sequent

$$\Gamma \vdash C$$

can be derived by one of the rules $(R\bot)$, $(L\otimes)$, $(R\otimes)$, or $(L\multimap)$.

<u>Case 1.</u> The sequent is derived by the rule $(R\bot)$.

In this case the derivation was the following:

$$\frac{\Gamma \vdash}{\Gamma \vdash \bot}\,(R\bot).$$

According to Lemma 3.1, we have $\Gamma = \bot$. Hence, the sequent $\Gamma \vdash \bot$ is an axiom: $\bot \vdash \bot$.

<u>Case 2.</u> The sequent is derived by the rule $(L\otimes)$.

In this case the derivation was the following:

$$\frac{A, B, \Pi \vdash C}{A \otimes B, \Pi \vdash C}\,(L\otimes).$$

The upper sequent is "good". Hence, according to the induction hypothesis it is derived without the $(L\bot)$ and $(R\bot)$ rules. Therefore, the lower sequent is derived without these rules too.

<u>Case 3.</u> The sequent is derived by the rule $(R\otimes)$.

In this case the derivation was the following:

$$\frac{\Pi \vdash A \qquad \Delta \vdash B}{\Pi, \Delta \vdash A \otimes B}\,(R\otimes).$$

Again both upper sequents are "good". Hence, according to the induction hypothesis it is derived without $(L\bot)$ and $(R\bot)$ rules. And therefore, the lower sequent is derived without these rules too.

Case 4. The sequent is derived by the rule $(L\multimap)$.
In this case the derivation was either the following:

$$\frac{\Pi \vdash A \qquad B, \Delta \vdash C}{A \multimap B, \Pi, \Delta \vdash C}\ (L\multimap),$$

or the following:

$$\frac{\Pi \vdash A, C \qquad B, \Delta \vdash}{A \multimap B, \Pi, \Delta \vdash C}\ (L\multimap).$$

The first case runs as before: both upper sequents are "good". Hence, they are derived without the $(L\bot)$ and $(R\bot)$ rules. And therefore, the lower sequent is derived without these rules too.

In the second case both upper sequents are "good". Therefore, according to Lemma 3.1 we have $B = \bot$, $\Delta = \Lambda$. But then the sequent

$$A \multimap B, \Pi, \Delta \vdash C$$

is not "good". We have got a contradiction. ■

Theorem 3.
$HLL_{[1]}(\bot)$ is NP-hard.
Proof.
Note, that if we shall substitute of constant $\bot$ for the literal q in the sequent (2) from Lemma 1, then the new sequent will be "good" (because of $b > 1$). Hence according to Lemma 3 the new sequent (with one literal and one constant) will be derivable if and only if the old one (with two literals) is derivable. So, according to Lemma 1 and NP-completeness of equipartition problem $HLL_{[1]}(\bot)$ is NP-hard. ■

Similarly, we immediately have a new proof of the NP-completness of $MLL_{[0]}(\bot)$.
Theorem 4.
$MLL_{[0]}(\bot)$ is NP-hard.
Proof.
The proof is similar to the proof of the last theorem. Note that if we substitute of constant $\bot$ for the literal q in the sequent (3) from Lemma 2, then we get a "good" sequent. Hence according to Lemma 3 the new sequent (with constant $\bot$) will be derivable if and only if the old one (with literal q) is derivable. So, according to Lemma 2 and NP-completeness of equipartition problem $MLL_{[0]}(\bot)$ is NP-hard. ■
Remark. Moreover we have proved, that the $MLL_{[0]}(\bot)$ without nested implication is NP-complete.

Corollary.
Both $MLL_{[0]}(\bot)$ and $HLL_{[1]}(\bot)$ are NP-complete.
Proof.
This follows from Theorems 3,4 and NP-completeness of MLL [Kanovich,1991].

■

2 Linear Time Decidability

2.1 One-literal Horn Fragments

Theorem 5.[Kanovich,1992]
$HLL_{[1]}(1)$ is decidable in linear time.

Theorem 6.[Kanovich,1992]
$HLLW_{[1]}(1)$ is decidable in linear time.

Theorem 7.
The Horn fragment of Linear Logic containing occurrences of only one literal and constants $\bot$ and 1 ($HLLW_{[1]}(\bot, 1)$) is decidable in linear time.
Proof.
The main idea of the proof is to exclude the constant $\bot$ from a sequent. In this case this fragment is reduced to $HLLW_{[1]}(1)$. And as it was shown (Theorem 6) this fragment is decidable in linear time.

A Horn sequent can contain the constant $\bot$ only in one of the following ways:

(a) $\bot \otimes B, \Gamma \vdash \Pi$;

(b) $(A \multimap \bot \otimes B), \Gamma \vdash \Pi$

(c) $\Gamma \vdash \bot \otimes B, \Pi$

(d) $(\bot \otimes B \multimap A), \Gamma \vdash \Pi$

Lemma 7.
Let Γ, Π be multisets of formulas, and let A, B be formulas (it is possible that B is the empty product). In this case LL with Weakening satisfies

(a) $\bot \otimes B, \Gamma \vdash \Pi$;

(b) $(A \multimap \bot \otimes B), \Gamma \vdash \Pi \Longleftrightarrow \Gamma \vdash A, \Pi$;

(c) $\Gamma \vdash \bot \otimes B, \Pi \iff \Gamma \vdash \Pi$;

(d) $(\bot \otimes B \multimap A), \Gamma \vdash \Pi \Longleftrightarrow \Gamma \vdash \Pi$.

Theorem 7 immediately follows from this lemma. Indeed, let us be given the sequent

$$\Gamma \vdash \Pi.$$

If Γ contains a formula of the form $\bot \otimes B$, then according to item (a) of Lemma 7 this sequent is derivable. In the other case, according to items (b), (c) and (d) of Lemma 7, we can exclude from this sequent all formulas containing $\bot$, moving from left to right. ■

Proof of Lemma 7.

(a) $\perp \otimes B, \Gamma \vdash \Pi$.

This item is evident:

$$\dfrac{\dfrac{\dfrac{\perp \vdash}{\perp, B \vdash}\,(W)}{\perp \otimes B \vdash}\,(R\otimes)}{\perp \otimes B, \Gamma \vdash \Pi}\,(W)$$

(b) $(A \multimap \perp \otimes B), \Gamma \vdash \Pi \Longleftrightarrow \Gamma \vdash A, \Pi$.

The first sequent is derived from the second one:

$$\dfrac{\Gamma \vdash A, \Pi \qquad \perp \otimes B \vdash}{(A \multimap \perp \otimes B), \Gamma \vdash \Pi}\;(L\multimap).$$

And the second sequent is derived from the first one:

$$\dfrac{(A \multimap \perp \otimes B), \Gamma \vdash \Pi \qquad \dfrac{\dfrac{A \vdash A}{A \vdash \perp \otimes B, A}\,(W)}{\vdash (A \multimap \perp \otimes B), A}\,(R\multimap)}{\Gamma \vdash A, \Pi}\,(CUT)$$

(c) $\Gamma \vdash \perp \otimes B, \Pi \Longleftrightarrow \Gamma \vdash \Pi$.

From right to left it is evident: the first sequent is derived from the second one by the Weakening rule.

And the second sequent is derived from the first one:

$$\dfrac{\Gamma \vdash \perp \otimes B, \Pi \qquad \perp \otimes B \vdash}{\Gamma \vdash \Pi}\;(CUT).$$

(d) $(\perp \otimes B \multimap A), \Gamma \vdash \Pi \Longleftrightarrow \Gamma \vdash \Pi$.

From right to left it is evident: the first sequent is derived from the second one by the Weakening rule.

And the second sequent is derived from the first one:

$$\dfrac{(\perp \otimes B \multimap A), \Gamma \vdash \Pi \qquad \dfrac{\dfrac{\perp \otimes B \vdash}{\perp \otimes B \vdash A}\,(W)}{\vdash \perp \otimes B \multimap A}\,(R\multimap)}{\Gamma \vdash \Pi}\,(CUT)$$

■

2.2 Multiplicative Fragment with Weakening and without Literal

Theorem 8.
$MLLW_{[0]}(\bot, 1)$ is decidable in linear time.
Proof.
To each formula and sequent α of $MLL_{[0]}(\bot, 1)$ we assign a number $\alpha^* \in \{0, 1\}$ in such a way that

$$1^* = 1$$

$$\bot^* = 0$$

$$(A \multimap B)^* = A^* \to B^*$$

$$(A \otimes B)^* = A^* \wedge B^*$$

$$(\Gamma \vdash \Delta)^* = \bigwedge_{A \in \Gamma} A^* \to \bigvee_{B \in \Delta} B^*$$

where $\to$ and $\wedge$ are classical implication and conjunction.

Lemma 8.
Sequent Φ is derivable in $MLLW_{[0]}(\bot, 1)$ if and only if $\Phi^* = 1$.

This Lemma is enough to prove the theorem. Really, Φ^* can be computed in linear time. Hence, $MLLW_{[0]}(\bot, 1)$ is decidable in linear time. ■

To prove Lemma 8 we shall prove Lemma 8.1.
Lemma 8.1.
For any formula A from $MLL_{[0]}(\bot, 1)$ it is true that

a) if $A^* = 1$, then the sequent $\vdash A$ is derivable in $MLLW_{[0]}(\bot, 1)$

b) if $A^* = 0$, then the sequent $A \vdash$ is derivable in $MLLW_{[0]}(\bot, 1)$

Proof.
We shall prove the lemma by induction on construction of formula A.
<u>Case 1.</u> Let $A = \bot$.
In this case sequent $A \vdash$ is an axiom: $\bot \vdash$.
<u>Case 2.</u> Let $A = 1$.
Similarly, in this case sequent $\vdash A$ is an axiom: $\vdash 1$.
<u>Case 3.</u> Let $A = B \otimes C$.
If $B^* = 0$, then $A^* = 0$ and we have got the derivation of the sequent $A \vdash$:

$$\cfrac{\cfrac{B \vdash \quad \text{(induction hypothesis)}}{B, C \vdash}\,(W)}{B \otimes C \vdash}\,(L\otimes)$$

The case when $C^* = 0$ is similar.

Now let $B^* = 1$ and $C^* = 1$. In this case $A^* = 1$ and we have got the derivation of the sequent $\vdash A$:

$$\frac{\vdash B \qquad\qquad \vdash C}{\vdash B \otimes C}\,(R\otimes) \quad \text{(induction hypotheses)}$$

Case 4. Let $A = B \multimap C$.

If $B^* = 0$, then $A^* = 1$, and we have got

$$\cfrac{\cfrac{B \vdash \quad \text{(induction hypothesis)}}{B \vdash C}\,(W)}{\vdash B \multimap C}\,(R\multimap)$$

If $C^* = 1$, then again $A^* = 1$, and we have got

$$\cfrac{\cfrac{\vdash C \quad \text{(induction hypothesis)}}{B \vdash C}\,(W)}{\vdash B \multimap C}\,(R\multimap)$$

Finally, if $B^* = 1$ and $C^* = 0$, then $A^* = 0$, and in this case

$$\frac{\vdash B \qquad\qquad C \vdash}{B \multimap C \vdash}\,(L\multimap) \quad \text{(induction hypotheses)}$$

■

Proof of Lemma 8.

If the sequent Φ is derivable in $MLLW_{[0]}(\bot, 1)$ then $\Phi^* = 1$. It is easy to prove this assertion by induction on derivation length.

We shall prove the reverse assertion.

If $(\Gamma \vdash \Delta)^* = 1$, then either there exists the formula $A \in \Gamma$, such that $A^* = 0$, or there exists the formula $B \in \Delta$, such that $B^* = 1$. According to Lemma 8.1 in the first case the sequent $A \vdash$ is derivable, and in the second case the sequent $\vdash B$ is derivable. In both cases, applying Weakening rule for several times, we shall get the derivable sequent $\Gamma \vdash \Delta$. ■

2.3 Horn Fragment without Literal

Theorem 9.

$HLL_{[0]}(\bot, 1)$ is decidable in linear time.

Proof.

Let us given the sequent $\Gamma \vdash \Delta$ from $HLL_{[0]}(\bot, 1)$.

Note, that $A \otimes 1 \stackrel{LL}{=\!=} A$, therefore we can assume that Δ consists of formulas of the form $\perp^a$, where $a \geq 0$ ($\perp^0 = 1$).

Lemma 9.1.

The sequent $\Gamma \vdash 1, \Delta$ is derivable in LL if and only if the sequent $\Gamma, \perp \vdash \Delta$ is derivable in LL.

Proof.

The sequent $\Gamma \vdash 1, \Delta$ is derived from the sequent $\Gamma, \perp \vdash \Delta$:

$$\dfrac{\Gamma, \perp \vdash \Delta \qquad \dfrac{\vdash 1}{\vdash 1, \perp}(R\perp)}{\Gamma \vdash 1, \Delta}(CUT)$$

Similarly, the sequent $\Gamma, \perp \vdash \Delta$ is derived from the sequent $\Gamma \vdash 1, \Delta$:

$$\dfrac{\Gamma \vdash 1, \Delta \qquad \dfrac{\perp \vdash}{\perp, 1 \vdash}(L1)}{\Gamma, \perp \vdash \Delta}(CUT)$$

■

According to this lemma we can consider without restriction of the generality that Δ contains only formulas of the form $\perp^a$, where $a \geq 1$.

Definition

If

$$\Delta = \perp^{a_1}, \perp^{a_2}, \ldots, \perp^{a_n},$$

then

$$\Delta^{\otimes} = \perp^{a_1+a_2+\ldots+a_n-n+1}.$$

(If $\Delta = \Lambda$, then $\Delta^{\otimes} = \perp$).

Lemma 9.2.

The sequent $\Delta^{\otimes} \vdash \Delta$ is derivable in LL.

Proof.

Let n be a number of formulas in Δ. We shall prove this Lemma by induction on n. If $n = 0$ then this sequent is an axiom: $\perp \vdash$. Suppose that the assertion is correct for $n = k$, and prove it for $n = k + 1$.

Let

$$\Delta = \Delta_0, \perp^a,$$

then

$$\Delta^{\otimes} = \Delta_0^{\otimes} \otimes \perp^{a-1}.$$

According to the induction hypothesis the sequent

$$\Delta_0^{\otimes} \vdash \Delta_0$$

is derivable. And the sequent $\Delta^{\otimes} \vdash \Delta$ is derived from this sequent:

$$\dfrac{\perp^{a-1} \vdash \perp^{a-1} \qquad \dfrac{\Delta_0^{\otimes} \vdash \Delta_0}{\Delta_0^{\otimes} \vdash \Delta_0, \perp}(R\perp)}{\Delta_0^{\otimes}, \perp^{a-1} \vdash \Delta_0, \perp^{a}}(R\otimes)$$

$$\dfrac{\Delta_0^{\otimes}, \perp^{a-1} \vdash \Delta_0, \perp^{a}}{\Delta_0^{\otimes} \otimes \perp^{a-1} \vdash \Delta_0, \perp^{a}}(L\otimes)$$

Thus we have proved that the sequent $\Delta^{\otimes} \vdash \Delta$ is derivable. ■

Lemma 9.3.
If

$$\Delta = \perp^{a_1}, \perp^{a_2}, \ldots, \perp^{a_n},$$

and Γ is the multiset of formulas from $HLL_{[0]}(\perp, 1)$, then the sequent $\Gamma \vdash \Delta$ is derivable in LL if and only if the sequent $\Gamma \vdash \Delta^{\otimes}$ is derivable in LL without the $(R\perp)$ and $(L\perp)$ rules. (i.e. the sequent $\Gamma \vdash \Delta$ is derivable in $HLL_{[0]}(\perp, 1)$ if and only if the sequent $\Gamma' \vdash \Delta^{\otimes\prime}$ is derivable in $HLL_{[1]}(1)$, where $'$ stands for substitution of a new literal f for constant $\perp$).

Proof.

The sequent $\Gamma \vdash \Delta$ is derived from the sequent $\Gamma \vdash \Delta^{\otimes}$:

$$\dfrac{\Gamma \vdash \Delta^{\otimes} \qquad \Delta^{\otimes} \vdash \Delta}{\Gamma \vdash \Delta}(CUT).$$

Now we prove the reverse assertion: if the sequent $\Gamma \vdash \Delta$ is derivable, then the sequent $\Gamma \vdash \Delta^{\otimes}$ is derivable, even without $(R\perp)$ and $(L\perp)$ rules.

We shall prove it by induction on derivation length of the sequent $\Gamma \vdash \Delta$.

This sequent can be derived by one of the rules $(R\otimes)$, $(L\multimap)$, $(L\perp)$, $(R\perp)$, $(L\otimes)$, $(L1)$ or this sequent is an axiom $A \vdash A$.

<u>Case 1.</u> The sequent is derived by the rule $(R\otimes)$.

In this case the derivation was the following:

$$\dfrac{\Gamma_1 \vdash \Delta_1, \perp^{a} \qquad \Gamma_2 \vdash \Delta_2, \perp^{b}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \perp^{a+b}}(R\otimes)$$

$(a \geq 1, b \geq 1)$.

Note, that

$$(\Delta_1, \perp^{a})^{\otimes} = \Delta_1^{\otimes} \otimes \perp^{a-1},$$

$$(\Delta_2, \perp^{b})^{\otimes} = \Delta_2^{\otimes} \otimes \perp^{b-1},$$

$$\Delta^{\otimes} = (\Delta_1, \Delta_2, \perp^{a+b})^{\otimes} = \Delta_1^{\otimes} \otimes \Delta_2^{\otimes} \otimes \perp^{a+b-2}.$$

According to the induction hypothesis, the following sequents are derivable:

$$\Gamma_1 \vdash \Delta_1^{\otimes} \otimes \perp^{a-1},$$

$$\Gamma_2 \vdash \Delta_2^{\otimes} \otimes \perp^{b-1}.$$

And the sequent $\Gamma \vdash \Delta^{\otimes}$ is derivable from these sequents:

$$\frac{\Gamma_1 \vdash \Delta_1^{\otimes} \otimes \perp^{a-1} \qquad \Gamma_1 \vdash \Delta_1^{\otimes} \otimes \perp^{a-1}}{\Gamma_1, \Gamma_2 \vdash \Delta_1^{\otimes} \otimes \Delta_2^{\otimes} \otimes \perp^{a+b-2}}\,(R\otimes).$$

<u>Case 2.</u> The sequent is derived by the rule $(L\multimap)$.
In this case the derivation was the following:

$$\frac{B, \Gamma_1 \vdash \Delta_1 \qquad \Gamma_2 \vdash A, \Delta_2}{A \multimap B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(L\multimap)$$

We can assume that $A \neq 1$, (because of $(1 \multimap B) \overset{LL}{=\!=} B$). Hence $A = \perp^a$, where $a \geq 1$. Let $\Delta_1^{\otimes} = \perp^{d_1}$, $\Delta_2^{\otimes} = \perp^{d_2}$, $(d_1, d_2 \geq 1)$, then $\Delta^{\otimes} = (\Delta_1, \Delta_2)^{\otimes} = \perp^{d_1+d_2-1}$. According to the induction hypothesis the following sequents are derivable:

$$B, \Gamma_1 \vdash \perp^{d_1},$$

$$\Gamma_2 \vdash \perp^{d_2+a-1}.$$

And the sequent $\Gamma \vdash \Delta^{\otimes}$ is derived from this sequent:

$$\frac{\dfrac{B,\Gamma_1 \vdash \perp^{d_1}}{\perp^{d_2-1} \otimes B, \Gamma_1 \vdash \perp^{d_1+d_2-1}}\,(M) \qquad \dfrac{\perp^{d_2+a-1}, \perp^a \multimap B \vdash \perp^{d_2-1} \otimes B \qquad \Gamma_2 \vdash \perp^{d_2+a-1}}{(\perp^a \multimap B), \Gamma_2 \vdash \perp^{d_2-1} \otimes B}\,(CUT)}{(\perp^a \multimap B), \Gamma_1, \Gamma_2 \vdash \perp^{d_1+d_2-1}}\,(CUT)$$

(Here (M) is the following rule:

$$\frac{X, \Gamma \vdash Y}{Z \otimes X, \Gamma \vdash Z \otimes Y}\,(M).$$

It is clear that this is an admissible rule).

<u>Case 3.</u> The sequent is derived by the rule $(L\perp)$.
In this case $\Gamma = \perp$, $\Delta = \Lambda$, $\Delta^{\otimes} = \perp$. Hence, the sequent $\Gamma \vdash \Delta^{\otimes}$ is an axiom: $\perp \vdash \perp$.

<u>Case 4.</u> The sequent is derived by the rule $(R\perp)$.
In this case the derivation was the following:

$$\frac{\Gamma \vdash \Delta_0}{\Gamma \vdash \Delta_0, \perp}\,(R\perp).$$

Then $\Delta^{\otimes} = (\Delta_0, \perp)^{\otimes} = \Delta_0^{\otimes}$. Hence, according to induction hypotheses the sequent $\Gamma \vdash \Delta^{\otimes}$ is derivable.

<u>Case 5.</u> The sequent is derived by the rule $(L\otimes)$.
In this case the derivation was the following:

$$\frac{A, B, \Gamma_0 \vdash \Delta}{A \otimes B, \Gamma_0 \vdash \Delta}\,(L\otimes).$$

Then the sequent $\Gamma \vdash \Delta^{\otimes}$ is derivable:

$$\dfrac{A, B, \Gamma_0 \vdash \Delta^{\otimes} \quad \text{(induction hypothesis)}}{A \otimes B, \Gamma_0 \vdash \Delta^{\otimes}} \, (L\otimes)$$

Case 6. The sequent is derived by the rule $(L1)$.
In this case the derivation was the following:

$$\dfrac{\Gamma_0 \vdash \Delta}{1, \Gamma_0 \vdash \Delta} \, (L1).$$

Then the sequent $\Gamma \vdash \Delta^{\otimes}$ is derivable:

$$\dfrac{\Gamma_0 \vdash \Delta^{\otimes} \quad \text{(induction hypothesis)}}{1, \Gamma_0 \vdash \Delta^{\otimes}} \, (L1)$$

Case 7. The sequent $\Gamma \vdash \Delta$ is an axiom $A \vdash A$.
In this case $\Gamma = \Delta = \perp^a$. Hence $\Delta^{\otimes} = \perp^a$ and the sequent $\Gamma \vdash \Delta^{\otimes}$ is an axiom $\perp^a \vdash \perp^a$
■

Using the last two lemmas it is possible to reduce the $HLL_{[0]}(\perp, 1)$ to $HLL_{[1]}(1)$ in linear time. Hence, according to Theorem 5 $HLL_{[0]}(\perp, 1)$ is decidable in linear time. ■

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