Decidability of Linear Affine Logic

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Abstract

The propositional Linear Logic is known to be undecidable. In the current paper we prove that the full propositional Linear Affine Logic containing all the multiplicatives, additives, exponentials, and constants is decidable. The proof is based on a reduction of Linear Affine Logic to sequents of specific "normal forms", and on a generalization of Kanovich computational interpretation of Linear Logic adapted to these "normal forms".

1 Introduction and Summary

Linear Logic has been introduced by Girard [1]. The undecidability of Linear Logic is established in [5]. Linear Affine Logic is Linear Logic with the weakening rule [6]. We will abbreviate Linear Logic and Linear Affine Logic as LL and LLW correspondingly. The decidability problem for Linear Affine Logic remained open.

In the current paper the decidability proof for Linear Affine Logic is based on the three results: firstly, the entire LLW is reduced (according to Turing) to its certain fragment (normal fragment). Secondly, the derivability of a normal form sequent is characterized in terms of vector games. Thirdly, by means of this computational interpretation, we prove that the normal fragment of LLW is decidable.

At the same time we prove that the entire Linear Logic is also reduced to its normal fragment and the derivability in this fragment is characterized in terms of analogous games.

2 Normal fragment and its computational interpretation

Now we give the definition of the normal fragment. The normal fragment is an expansion of the $\oplus$-Horn fragment [2, 3]. Let us recall some definitions from [2].

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Definitions *A simple product* is a tensor product of literals and the constant 1. (For example: 1, p, $p \otimes q$). *A Horn implication* is an implication of the form: $X \multimap Y$, where X and Y are simple products. *A $\oplus$-Horn implication* is an implication of the form: $X \multimap (Y \oplus Z)$, where X, Y and Z are simple products.

Definitions *A simple disjunction* is a disjunction of the form: $X \wp Y$, where X and Y are simple products. Horn implications, $\oplus$-Horn implications, and simple disjunctions are called *normal formulas*. *A normal sequent* is a sequent of the form

$$W, !\Gamma \Rightarrow ?\Delta,$$

where W is a simple product, Γ is a multiset of normal formulas, and Δ is a multiset of simple products. *The normal fragment* is a fragment containing only normal sequents.

Let us consider the computational interpretation of normal sequents. There are two view-points on the interpretation of a sequent of the form $W, !\Gamma \Rightarrow ?\Delta$ From the first point of view (more traditional), Γ is interpreted as a set of operations with simple products. In this case, the sequent $W, !\Gamma \Rightarrow ?\Delta$ is interpreted as a proposition, that we can obtain a set of simple products contained in Δ from the simple product W (see Figure 1a).

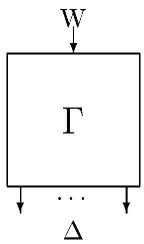

Figure 1a.

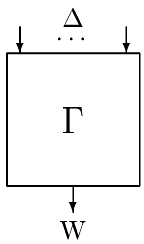

Figure 1b.

The same approach was used in [3]. That paper deals with $\oplus$-Horn sequents, i.e. sequents of the form

$$
\begin{array}{ll}
(I) & \overline{W \Rightarrow W} \\
(L\otimes) & \dfrac{W,\Gamma \Rightarrow \Delta}{U,\Gamma \Rightarrow \Delta}, \quad \text{where } \vec{W} = \vec{U}. \\
(H) & \dfrac{Y \otimes U, \Gamma \Rightarrow \Delta}{X \otimes U, X \multimap Y, \Gamma \Rightarrow \Delta} \\
(H\oplus) & \dfrac{Y_1 \otimes U, \Gamma \Rightarrow \Delta \qquad Y_2 \otimes U, \Gamma \Rightarrow \Delta}{X \otimes U, X \multimap (Y_1 \oplus Y_2), \Gamma \Rightarrow \Delta} \\
(H\wp) & \dfrac{Y_1 \otimes U_1, \Gamma_1 \Rightarrow \Delta_1 \qquad Y_2 \otimes U_2, \Gamma_2 \Rightarrow \Delta_2}{U_1 \otimes U_2, Y_1 \wp Y_2, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \\
(CUT) & \dfrac{W, \Gamma_1 \Rightarrow \Delta_1, U \qquad U, \Gamma_2 \Rightarrow \Delta_2}{W, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \\
(M) & \dfrac{W, \Gamma \Rightarrow Z, \Delta}{W \otimes V, \Gamma \Rightarrow Z \otimes V, \Delta} \\
(L!) \ \dfrac{W, A, \Gamma \Rightarrow \Delta}{W, !A, \Gamma \Rightarrow \Delta} & (R?) \ \dfrac{W, \Gamma \Rightarrow \Delta, Z}{W, \Gamma \Rightarrow \Delta, ?Z} \\
(W!) \ \dfrac{W, \Gamma \Rightarrow \Delta}{W, !A, \Gamma \Rightarrow \Delta} & (W?) \ \dfrac{W, \Gamma \Rightarrow \Delta}{W, \Gamma \Rightarrow \Delta, ?Z} \\
(C!) \ \dfrac{W, !A, !A, \Gamma \Rightarrow \Delta}{W, !A, \Gamma \Rightarrow \Delta} & (C?) \ \dfrac{W, \Gamma \Rightarrow \Delta, ?Z, ?Z}{W, \Gamma \Rightarrow \Delta, ?Z}
\end{array}
$$

Table 1a: The inference rules of NLL.

$$
(W) \quad \frac{W, \Gamma \Rightarrow \Delta}{W \otimes U, \Gamma \Rightarrow \Delta}; \qquad \frac{W, \Gamma_0 \Rightarrow \Delta_0}{W, \Gamma \Rightarrow \Delta}, \qquad \text{where } \Gamma_0 \subseteq \Gamma, \Delta_0 \subseteq \Delta
$$

Table 1b: The weakening rules.

$W, !\Gamma \Rightarrow Z$, where W and Z are simple products and Γ is a list of Horn and $\oplus$-Horn implications. In [3] Γ is interpreted as a non-determinate Petri net, W and Z are interpreted as markings, such that the sequent $W, !\Gamma \Rightarrow Z$ is derivable in LL if and only if the marking Z is reachable from the marking W.

From the second point of view, Γ is interpreted as a list of inversion operations, and the sequent $W, !\Gamma \Rightarrow ?\Delta$ is interpreted as a proposition, that we can obtain the simple product W from a set of simple products Δ (see Figure 1b).

Let us give the rigorous definition of this interpretation of a normal sequent. We begin with the remark: there is a simple correspondence between simple products and vectors with natural coordinates (cf. [3]). Namely, we can associate a n-dimensional vector $(x_1, x_2, \ldots, x_n)$ with a simple product $p_1^{x_1} \otimes p_2^{x_2} \otimes \ldots \otimes p_n^{x_n}$. Here $p^x = p \otimes p \otimes \ldots \otimes p$ (x times) and $p^0 = 1$. If X is a simple product, then $\vec{X}$ denotes the corresponding vector.

Let $\Phi = (W, !\Gamma \Rightarrow ?\Delta)$ be a normal sequent. Let us consider the following two single-person games associated with this sequent.

Game $\mathbf{A}_\Phi$

1. Initially, all vectors from $\vec{\Delta}$ are written on the blackboard.
2. We may write new vectors *with natural coordinates* by the following rules:
   (a) If $X \multimap Y \in \Gamma$ and a vector $a + \vec{Y}$ has been already written, then we may write $a + \vec{X}$.
   (b) If $X \multimap (Y_1 \oplus Y_2) \in \Gamma$ and vectors $a + \vec{Y_1}$ and $a + \vec{Y_2}$ have been already written, then we may write $a + \vec{X}$.
   (c) If $Y_1 \wp Y_2 \in \Gamma$ and vectors $a_1 + \vec{Y_1}$ and $a_2 + \vec{Y_2}$ have been already written, then we may write $a_1 + a_2$.
3. The aim of the game is to obtain the vector $\vec{W}$.

Game $\mathbf{B}_\Phi$ This game is the game $\mathbf{A}_\Phi$ with the additional rule:

4. If a vector a has been written and $a \leq c$ then[1] we may write c.

In the current section we prove the following

Theorem 1 (The computational interpretation) *It is possible to reach the aim in the game $\mathbf{A}_\Phi$ ($\mathbf{B}_\Phi$), if and only if the sequent Φ is derivable in LL (LLW).*

Proof To prove the theorem, we introduce auxiliary logics NLL and NLLW (Normal Linear Logic and Normal Linear Logic with the Weakening rule). These logics are the development of calculus HLL, which was

[1]We say that $(x_1 \ldots x_n) \leq (y_1 \ldots y_n)$ iff $\forall i\,(x_i \leq y_i)$.

used in [2] for justification of the computational interpretation of the $\oplus$-Horn fragment of Linear Logic. The inference rules of NLL are represented in Table 1a. And NLLW is NLL with the weakening rules (Table 1b).

In these tables U, V, W, X, Y, Z are simple products; $\Gamma, \Gamma_1, \Gamma_2$ are multisets of normal formulas and normal formulas with "!"; $\Delta, \Delta_1, \Delta_2$ are multisets of simple products and simple products with "?"; and A is a normal formula. We define the NLL and NLLW so that the set of normal sequents which are derivable in NLL (NLLW) is equal to the set of normal sequents which are derivable in LL (LLW). In order to verify it we use the following four lemmas.

Lemma 1.1 *If $\Pi = P_1 \ldots P_n$ is a multiset of simple products and the sequent*

$$\Pi, \Gamma \Rightarrow \Delta$$

is derivable in LL (LLW), then the sequent

$$\Pi^{\otimes}, \Gamma \Rightarrow \Delta$$

is derivable in NLL (NLLW), where $\Pi^{\otimes} = P_1 \otimes \ldots \otimes P_n$.

Lemma 1.2 *We can eliminate the rule (M) both in NLL and NLLW.*

Lemma 1.3 *Let the sequent*

$$W, \Gamma, !\tilde{\Gamma} \Rightarrow \Delta, ?\tilde{\Delta}$$

be derivable in NLL (NLLW) without the rule (M). Here $\Gamma, \tilde{\Gamma}$ are multisets of normal formulas and $\Delta, \tilde{\Delta}$ are multisets of simple products. If $\Phi = (W, !\Gamma, !\tilde{\Gamma} \Rightarrow ?\Delta, ?\tilde{\Delta})$, then it is possible to reach the aim in the game $\mathbf{A}_{\Phi}$ $(\mathbf{B}_{\Phi})$.

Lemma 1.4 *Let Φ be a normal sequent. Assume that one can write the vector $\vec{W}$ by the rules of the game $\mathbf{A}_{\Phi}$ $(\mathbf{B}_{\Phi})$. Then the sequent Φ is derivable in LL (LLW).*

It is easy to prove all these lemmas by induction on the derivation. And it follows from these lemmas that the following three assertions are equivalent:

- $LL \vdash \Phi$,
- $NLL \vdash \Phi$,
- It is possible to reach the aim in the game $\mathbf{A}_{\Phi}$.

And by analogy for the Affine Logic the following three assertions are equivalent:

- $LLW \vdash \Phi$,
- $NLLW \vdash \Phi$,
- It is possible to reach the aim in the game $\mathbf{B}_{\Phi}$.

Hence, Theorem 1 is proved. ■

3 The decidability of the normal fragment of LLW

Theorem 2 *The normal fragment of Linear Affine Logic is decidable.*

We begin with

Definitions 1. The set of vectors $\mathcal{A}$ is closed under the Horn implication $X \multimap Y$ if

$$\forall a \in \omega^n \quad a + \vec{Y} \in \mathcal{A} \Rightarrow a + \vec{X} \in \mathcal{A}.$$

(Here ω^n is the set of vectors with natural coordinates).

2. The set of vectors $\mathcal{A}$ is closed under the $\oplus$-Horn implication $X \multimap (Y_1 \oplus Y_2)$ if

$$\forall a \in \omega^n \quad a + \vec{Y_1} \in \mathcal{A}, a + \vec{Y_2} \in \mathcal{A} \Rightarrow a + \vec{X} \in \mathcal{A}.$$

3. The set of vectors $\mathcal{A}$ is closed under the simple disjunction $Y_1 \wp Y_2$ if

$$\forall a_1, a_2 \in \omega^n \quad a_1 + \vec{Y_1} \in \mathcal{A}, a_2 + \vec{Y_2} \in \mathcal{A} \Rightarrow a_1 + a_2 \in \mathcal{A}$$

4. Let Γ be a multiset of normal formulas. The set of vectors $\mathcal{A}$ is Γ-closed if it is closed under all formulas from Γ.

5. The set of vectors $\mathcal{A}$ is closed under the weakening rule if

$$\forall a \in \mathcal{A} \ \forall c \geq a \quad c \in \mathcal{A}.$$

Lemma 2.1 *Let $\Phi = (W, !\Gamma \vdash ?\Delta)$ be a normal sequent. Then the following two conditions are equivalent:*

(i) It is possible to reach the aim in the game $\mathbf{B}_{\Phi}$.

(ii) For any $\mathcal{A} \subseteq \omega^n$ if $\mathcal{A}$ is Γ-closed and closed under the weakening rule and $\vec{\Delta} \subseteq \mathcal{A}$ then $\vec{W} \in \mathcal{A}$.

Proof

(ii)$\Longrightarrow$(i) Let us consider the set

$$\mathcal{A} = \{x : x \text{ can be obtained by the rules of } \mathbf{B}_{\Phi}\}.$$

By the definition of the game $\mathbf{B}_{\Phi}$ and closed sets, the set $\mathcal{A}$ is Γ-closed and closed under the weakening rule. Moreover, it includes the set $\vec{\Delta}$. Hence, according to our hypothesis, $\vec{W} \in \mathcal{A}$. So, $\vec{W}$ can be obtained by the rules of the game $\mathbf{B}_{\Phi}$.

(i)$\Longrightarrow$(ii) Let $\mathcal{A}$ be any Γ-closed set which is closed under the weakening rule, and $\vec{\Delta} \subseteq \mathcal{A}$. Then, by the definition of the game $\mathbf{B}_\Phi$ and closed sets, $\mathcal{A}$ includes all vectors, which can be written by the rules of the game $\mathbf{B}_\Phi$. In particular, $\vec{W} \in \mathcal{A}$. ∎

Remark It is possible to formulate the analogous lemma for the game $\mathbf{A}_\Phi$.

Definitions *A cone with the vertex z* is a set of vectors $K_z = \{y : y \geq z\}$.

Note, that any set of vectors, which is closed under the weakening rule is a union of cones. We shall prove that such set is a union of a *finite* set of cones. This fact follows from the central lemma:

Lemma 2.2 *Any set of vectors from ω^n which are incomparable pairwise is finite.*

Proof Let $\mathcal{A} \subseteq \omega^n$ and $\forall x, y \in \mathcal{A}$ $(x \not< y)$. We shall prove that $\mathcal{A}$ is finite by induction on n. For $n = 1$ it is trivial. To verify the induction step, we consider any vector $a \in A$. Let $a = (a_1, \ldots, a_n)$. We define the following sets:

$$\mathcal{A}_i = \{x \in \mathcal{A} : x_i \leq a_i\}.$$

One can easily verify that $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$. Now we define the sets

$$\mathcal{A}_{ij} = \{x \in \mathcal{A}_i : x_i = j\}.$$

It is evident, that $\mathcal{A}_i = \bigcup_{j=1}^{a_i} \mathcal{A}_{ij}$. By the induction hypothesis, the sets $\mathcal{A}_{ij}$ are finite. Hence, $\mathcal{A}$ is finite too. ∎

Corollary 2.3 *Let $\mathcal{A}$ be a set of vectors with natural coordinates. In this case if $\mathcal{A}$ is closed under the weakening rule, then $\mathcal{A}$ is a union of finite set of cones.*

Proof Let $\mathcal{B}$ be a set of the minimal elements of the set $\mathcal{A}$, i.e.

$$\mathcal{B} = \{z \in \mathcal{A} : \neg\exists y \in \mathcal{A} \quad y < z\}.$$

Then

$$\mathcal{A} = \bigcup_{z \in \mathcal{B}} K_z.$$

According to 2.2, $\mathcal{B}$ is finite. ∎

Definition Let $x, y \in \omega^n$, then

$$max(x, y) := (max(x_1, y_1), \ldots, max(x_n, y_n)).$$

Definition Let $\mathcal{A}, \mathcal{B} \subseteq \mathbf{Z}^n$, $x \in \mathbf{Z}^n$ (here $\mathbf{Z}$ is a set of integers), then

$$\mathcal{A} + x := \{a + x : a \in \mathcal{A}\},$$

$$\mathcal{A} + \mathcal{B} := \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Lemma 2.4 *Let $\mathcal{A} = \bigcup_{z \in \mathcal{B}} K_z$, then*

1. *$\mathcal{A}$ is closed under $X \multimap Y$ if and only if*

$$\forall z \in \mathcal{B} \quad max(z - \vec{Y}, 0) + \vec{X} \in \mathcal{A}.$$

2. *$\mathcal{A}$ is closed under $X \multimap (Y_1 \oplus Y_2)$ if and only if*

$$\forall z_1, z_2 \in \mathcal{B} \quad max(z_1 - \vec{Y}_1, z_2 - \vec{Y}_2, 0) + \vec{X} \in \mathcal{A}.$$

3. *$\mathcal{A}$ is closed under $Y_1 \wp Y_2$ if and only if*

$$\forall z_1, z_2 \in \mathcal{B} \quad max(z_1 - \vec{Y}_1, 0) + max(z_2 - \vec{Y}_2, 0) \in \mathcal{A}$$

Proof 1. By the definition, $\mathcal{A}$ is closed under $X \multimap Y$ if and only if

$$\{a + \vec{X} : a \in w^n; a + \vec{Y} \in \mathcal{A}\} \subseteq \mathcal{A}.$$

But
$\{a + \vec{X} : a \in w^n; a + \vec{Y} \in \mathcal{A}\} = (\mathcal{A} - \vec{Y}) \cap \omega^n + \vec{X} =$
$(\bigcup\limits_{z \in \mathcal{B}} K_z - \vec{Y}) \cap \omega^n + \vec{X} = \bigcup\limits_{z \in \mathcal{B}} [(K_z - \vec{Y}) \cap \omega^n + \vec{X}] =$
$\bigcup\limits_{z \in \mathcal{B}} [K_{z - \vec{Y}} \cap \omega^n + \vec{X}] = \bigcup\limits_{z \in \mathcal{B}} [K_{max(z - \vec{Y}, 0)} + \vec{X}] =$
$\bigcup\limits_{z \in \mathcal{B}} K_{max(z - \vec{Y}, 0) + \vec{X}}$.

So, $\mathcal{A}$ is closed under $X \multimap Y$ if and only if

$$\bigcup_{z \in \mathcal{B}} K_{max(z - \vec{Y}, 0) + \vec{X}} \subseteq \mathcal{A},$$

i.e.

$$\forall z \in \mathcal{B} \quad max(z - \vec{Y}, 0) + \vec{X} \in \mathcal{A}.$$

2. The situation is similar to the previous item. $\mathcal{A}$ is closed under $X \multimap (Y_1 \oplus Y_2)$ if and only if

$$\{a + \vec{X} : a \in w^n; a + \vec{Y}_1 \in \mathcal{A}; a + \vec{Y}_2 \in \mathcal{A}\} \subseteq \mathcal{A}.$$

But
$\{a + \vec{X} : a \in w^n; a + \vec{Y}_1 \in \mathcal{A}; a + \vec{Y}_2 \in \mathcal{A}\} =$
$(\mathcal{A} - \vec{Y}_1) \cap (\mathcal{A} - \vec{Y}_2) \cap \omega^n + \vec{X} =$
$(\bigcup\limits_{z \in \mathcal{B}} K_z - \vec{Y}_1) \cap (\bigcup\limits_{z \in \mathcal{B}} K_z - \vec{Y}_2) \cap \omega^n + \vec{X} =$
$(\bigcup\limits_{z \in \mathcal{B}} K_{z - \vec{Y}_1}) \cap (\bigcup\limits_{z \in \mathcal{B}} K_{z - \vec{Y}_2}) \cap \omega^n + \vec{X} =$

$$\dfrac{\dfrac{!\Gamma, A \Rightarrow B}{!\Gamma, !A \Rightarrow B}(L!)}{!\Gamma, !A \Rightarrow !B}(R!)$$

Proof of $!A \overset{\Gamma}{\simeq} !B$

$$\dfrac{\dfrac{!\Gamma, A \Rightarrow B}{!\Gamma, A \Rightarrow B \oplus C}(R\oplus) \qquad \dfrac{C \Rightarrow C}{!\Gamma, C \Rightarrow B \oplus C}(R\oplus, W!)}{!\Gamma, A \oplus C \Rightarrow B \oplus C}(L\oplus)$$

Proof of $A \oplus C \overset{\Gamma}{\simeq} B \oplus C$

$$\dfrac{\dfrac{!\Gamma, A \Rightarrow B \qquad C \Rightarrow C}{!\Gamma, C, C \multimap A \Rightarrow B}(L\multimap)}{!\Gamma, C \multimap A \Rightarrow C \multimap B}(R\multimap)$$

Proof of $C \multimap A \overset{\Gamma}{\simeq} C \multimap B$

$$\dfrac{\dfrac{!\Gamma, A \Rightarrow B}{!\Gamma, A \Rightarrow ?B}(R?)}{!\Gamma, ?A \Rightarrow ?B}(L?)$$

Proof of $?A \overset{\Gamma}{\simeq} ?B$

$$\dfrac{\dfrac{!\Gamma, A \Rightarrow B}{!\Gamma, A\&C \Rightarrow B}(L\&) \qquad \dfrac{C \Rightarrow C}{!\Gamma, A\&C \Rightarrow C}(L\&, W!)}{!\Gamma, A\&C \Rightarrow B\&C}(L\&)$$

Proof of $A\&C \overset{\Gamma}{\simeq} B\&C$

$$\dfrac{\dfrac{!\Gamma, B \Rightarrow A \qquad C \Rightarrow C}{!\Gamma, B, A \multimap C \Rightarrow C}(L\multimap)}{!\Gamma, A \multimap C \Rightarrow B \multimap C}(R\multimap)$$

Proof of $A \multimap C \overset{\Gamma}{\simeq} B \multimap C$

$$\dfrac{\dfrac{!\Gamma, B \Rightarrow A}{!\Gamma, A^{\perp}, B \Rightarrow}(L^{\perp})}{!\Gamma, A^{\perp} \Rightarrow B^{\perp}}(R^{\perp})$$

Proof of $A^{\perp} \overset{\Gamma}{\simeq} B^{\perp}$

$$\dfrac{\dfrac{!\Gamma, A \Rightarrow B \qquad C \Rightarrow C}{!\Gamma, A, C \Rightarrow B \otimes C}(R\otimes)}{!\Gamma, A \otimes C \Rightarrow B \otimes C}(L\otimes)$$

Proof of $A \otimes C \overset{\Gamma}{\simeq} B \otimes C$

$$\dfrac{\dfrac{!\Gamma, A \Rightarrow B \qquad C \Rightarrow C}{!\Gamma, A\wp C \Rightarrow B, C}(L\wp)}{!\Gamma, A\wp C \Rightarrow B\wp C}(R\wp)$$

Proof of $A\wp C \overset{\Gamma}{\simeq} B\wp C$

Table 2: The proof of Lemma 3.1.1

$\bigcup_{z_1,z_2\in\mathcal{B}} K_{z_1-\vec{Y}_1} \cap K_{z_2-\vec{Y}_2} \cap \omega^n + \vec{X} =$
$\bigcup_{z_1,z_2\in\mathcal{B}} K_{max(z_1-\vec{Y}_1, z_2-\vec{Y}_2,0)+\vec{X}}$.

So, $\mathcal{A}$ is closed under $X \multimap (Y_1 \oplus Y_2)$ if and only if

$$\bigcup_{z_1,z_2\in\mathcal{B}} K_{max(z_1-\vec{Y}_1, z_2-\vec{Y}_2,0)+\vec{X}} \subseteq \mathcal{A},$$

i.e.

$$\forall z_1, z_2 \in \mathcal{B} \quad max(z_1 - \vec{Y}_1, z_2 - \vec{Y}_2, 0) + \vec{X} \in \mathcal{A}.$$

3. $\mathcal{A}$ is closed under $Y_1 \wp Y_2$ if and only if

$$\{a_1 + a_2 : a_1, a_2 \in w^n; a_1 + \vec{Y}_1 \in \mathcal{A}; a_2 + \vec{Y}_2 \in \mathcal{A}\} \subseteq \mathcal{A}.$$

But
$\{a_1 + a_2 : a_1, a_2 \in w^n; a_1 + \vec{Y}_1 \in \mathcal{A}; a_2 + \vec{Y}_2 \in \mathcal{A}\} =$
$(\mathcal{A} - \vec{Y}_1) \cap \omega^n + (\mathcal{A} - \vec{Y}_2) \cap \omega^n =$
$(\bigcup_{z\in\mathcal{B}} K_z - \vec{Y}_1) \cap \omega^n + (\bigcup_{z\in\mathcal{B}} K_z - \vec{Y}_2) \cap \omega^n =$
$\bigcup_{z\in\mathcal{B}} K_{max(z-\vec{Y}_1,0)} + \bigcup_{z\in\mathcal{B}} K_{max(z-\vec{Y}_2,0)} =$
$\bigcup_{z_1,z_2\in\mathcal{B}} K_{max(z_1-\vec{Y}_1,0)+max(z_2-\vec{Y}_2,0)}$.

So, $\mathcal{A}$ is closed under $Y_1 \wp Y_2$ if and only if

$$\bigcup_{z_1,z_2\in\mathcal{B}} K_{max(z_1-\vec{Y}_1,0)+max(z_2-\vec{Y}_2,0)} \subseteq \mathcal{A},$$

i.e.

$$\forall z_1, z_2 \in \mathcal{B} \quad max(z_1 - \vec{Y}_1, 0) + max(z_2 - \vec{Y}_2, 0) \in \mathcal{A}.\ \blacksquare$$

Corollary 2.5 *For any finite set of vectors $\mathcal{B}$ the property of the set $\mathcal{A} = \bigcup_{z\in\mathcal{B}} K_z$ to be Γ-closed is decidable.*

Proof Note that all conditions in the last lemma are decidable. Hence, the property to be Γ-closed is also decidable. $\blacksquare$

Now we can prove Theorem 2. Let us look at the condition (ii) from Lemma 2.1. By Corollary 2.3 this condition is equivalent to the following condition:

(iii) For any finite set $\mathcal{B} \subset \omega^n$ if $\bigcup_{z\in\mathcal{B}} K_z$ is Γ-closed and $\vec{\Delta} \subseteq \bigcup_{z\in\mathcal{B}} K_z$ then $\vec{W} \in \bigcup_{z\in\mathcal{B}} K_z$.

By Corollary 2.5 the condition standing after the words "For any finite set $\mathcal{B} \subset \omega^n \ldots$" is decidable. Hence, the condition (iii) is recursively co-enumerable. By Lemma 2.1 the condition (iii) is equivalent to the condition (i). And by Theorem 1 the condition (i), in its turn, is equivalent to the condition, that $LLW \vdash \Phi$. Hence, the set of derivable normal sequents in LLW is co-enumerable. On the other hand it is clear that this set is recursively enumerable. So, by Post's Theorem, it is decidable. $\blacksquare$

4 Reduction to the normal fragment

Theorem 3 *Linear Logic and Linear Affine Logic are reduced (according to Turing) to their normal fragments.*

Definition A *"good"* sequent is a sequent of the form: $\Lambda, !\Gamma \vdash \Delta$, where Λ, Γ and Δ contain neither "!" nor "?".

The proof of Theorem 3 consists of two steps. Firstly, we prove that the "good" fragment is reduced to the normal fragment (Lemma 3.1). And secondly, we prove that the whole LL and LLW are reduced to their "good" fragment (Lemma 3.2).

Lemma 3.1 *For any "good" sequent Φ one can effectively construct a normal sequent Ψ such that*

$$\begin{array}{rcl} LLW \vdash \Phi & \textit{iff} & LLW \vdash \Psi; \\ LL \vdash \Phi & \textit{iff} & LL \vdash \Psi. \end{array}$$

In order to prove Lemma 3.1 we introduce

Definition Let A and B be formulas, and Γ be a multiset of formulas. We say that $A \overset{\Gamma}{\simeq} B$ in any logic (A is equal to B according to Γ), if the following sequents are derivable in the logic:

$$\begin{array}{l} !\Gamma, A \Rightarrow B, \\ !\Gamma, B \Rightarrow A. \end{array}$$

If Γ is an empty multiset, then we write $A \simeq B$ (it is a traditional equality).

By analogous to the traditional equality the following lemma is holds for LL as well as for LLW:

Lemma 3.1.1 *If $A \overset{\Gamma}{\simeq} B$ then*

$$\begin{array}{lll} !A \overset{\Gamma}{\simeq} !B, & A \oplus C \overset{\Gamma}{\simeq} B \oplus C, & C \multimap A \overset{\Gamma}{\simeq} C \multimap B, \\ ?A \overset{\Gamma}{\simeq} ?B, & A \& C \overset{\Gamma}{\simeq} B \& C, & A \multimap C \overset{\Gamma}{\simeq} B \multimap C, \\ A^{\perp} \overset{\Gamma}{\simeq} B^{\perp}, & A \otimes C \overset{\Gamma}{\simeq} B \otimes C, \textit{ and } & A \wp C \overset{\Gamma}{\simeq} B \wp C. \end{array}$$

Proof Let the sequents

$$!\Gamma, A \Rightarrow B, \qquad !\Gamma, B \Rightarrow A$$

be derivable. We must prove the derivability of the sequents of the form $!\Gamma, D(A) \Rightarrow D(B)$ and of the form $!\Gamma, D(B) \Rightarrow D(A)$, where $D(p) = !p$, $p \oplus C$, $C \multimap p$, and so on. The derivations of the first sequents are represented in Table 2. The derivations of the second sequents are obtained by replacement A by B and B by A in the derivations in Table 2. ∎

Lemma 3.1.2 *If $A \overset{\Gamma}{\simeq} B$ then the sequent $!\Gamma, \Pi \Rightarrow \Delta$ is derivable if and only if the sequent $!\Gamma, \Pi' \Rightarrow \Delta'$ is derivable, where $'$ stands for substitution of the formula B for some occurrences of the formula A.*

Proof We may assume without loss of generality that $'$ stands for substitution of formula B for only one occurrence of formula A. Let us suppose that formula A occurs in the external formula $C = C(A)$ (i.e., $C \in \Pi$ or $C \in \Delta$). Then $C' = C(B)$ and according to the last lemma $C \overset{G}{\simeq} C'$.

Case 1. $C \in \Pi$, i.e. $\Pi = C, \Pi_0$; $\Pi' = C', \Pi_0$; $\Delta' = \Delta$.

Then the sequent $!\Gamma, \Pi' \Rightarrow \Delta'$ is derivable from the sequent $!\Gamma, \Pi \Rightarrow \Delta$:

$$\dfrac{\dfrac{!\Gamma, C, \Pi_0 \Rightarrow \Delta \qquad !\Gamma, C' \Rightarrow C}{!\Gamma, !\Gamma, C', \Pi_0 \Rightarrow \Delta}{\scriptstyle (CUT)}}{!\Gamma, C', \Pi_0 \Rightarrow \Delta}{\scriptstyle (C!)} \tag{1}$$

Case 2. $C \in \Delta$, i.e. $\Delta = C, \Delta_0$; $\Delta' = C', \Delta_0$; $\Pi' = \Pi$.

Then the sequent $!\Gamma, \Pi' \Rightarrow \Delta'$ is derivable from the sequent $!\Gamma, \Pi \Rightarrow \Delta$:

$$\dfrac{\dfrac{!\Gamma, \Pi \Rightarrow C, \Delta_0 \qquad !\Gamma, C \Rightarrow C'}{!\Gamma, !\Gamma, \Pi \Rightarrow C', \Delta_0}{\scriptstyle (CUT)}}{!\Gamma, \Pi \Rightarrow C', \Delta_0}{\scriptstyle (C!)} \tag{2}$$

Replace the formula C by the formula C' and the formula C' by the formula C in the derivations (1) and (2). We obtain the derivation of the sequent $!\Gamma, \Pi \Rightarrow \Delta$ from the sequent $!\Gamma, \Pi' \Rightarrow \Delta'$. ∎

Lemma 3.1.3 *The sequent*

$$!(A \multimap p), !(p \multimap A), \Pi \Rightarrow \Delta$$

is derivable if and only if the sequent

$$\Pi' \Rightarrow \Delta'$$

is derivable, where $'$ stands for substitution of the formula A for all *occurrences of the literal p.*

Proof Let

$$\Gamma = A \multimap p, p \multimap A.$$

Then $A \overset{\Gamma}{\simeq} p$. Hence, if the sequent $\Pi' \Rightarrow \Delta'$ is derivable, then the sequent $!\Gamma, \Pi \Rightarrow \Delta$ is derivable too (by the rule $(W!)$ and the last lemma).

Conversely, suppose that the sequent $!(A \multimap p), !(p \multimap A), \Pi \Rightarrow \Delta$ is derivable. If we substitute the formula A for the literal p in this sequent, then we obtain the derivable sequent:

$$!(A \multimap A), !(A \multimap A), \Pi' \Rightarrow \Delta'.$$

The sequent $\Pi' \Rightarrow \Delta'$ is derivable from this sequent:

$$\dfrac{\dfrac{!(A \multimap A), !(A \multimap A), \Pi' \Rightarrow \Delta'}{!(A \multimap A), \Pi' \Rightarrow \Delta'}{\scriptstyle (C!)} \qquad \Rightarrow !(A \multimap A)}{\Pi' \Rightarrow \Delta'}{\scriptstyle (CUT)}$$

∎

Using this lemma, let us prove Lemma 3.1 for LLW. Let we have a "good" sequent Φ. We shall transform this sequent, watching that the new sequent will be always derivable in LLW if and only if the old one is derivable too.

Firstly, by De Morgan equalities, we eliminate from our sequent the following connectives: &, $\wp$, $\multimap$, $\bot$, 0. Namely, we replace the formulas $A\&B$ by $(A^\bot \oplus B^\bot)^\bot$, $A\wp B$ by $(A^\bot \otimes B^\bot)^\bot$, $A \multimap B$ by $(A \otimes B^\bot)^\bot$, $\bot$ by $1^\bot$, and 0 by $\top^\bot$.

Secondly, using that $1 \simeq \top$ *in LLW*, we replace $\top$ by 1. After that we obtain a "good" sequent Φ_1 which contains only the following connectives: !, $\otimes$, $\oplus$, $^\bot$, and 1.

Then, using Lemma 3.1.3, we may replace all formulas of the forms $p \otimes q$, $p \oplus q$, $p^\bot$ and 1 by new literals and add formulas of the form from Table 3 to the antecedent.

$!(r \multimap (p \otimes q))$,	$!((p \otimes q) \multimap r)$,
$!(r \multimap (p \oplus q))$,	$!((p \oplus q) \multimap r)$,
$!(r \multimap p^\bot)$,	$!(p^\bot \multimap r)$,
$!(r \multimap 1)$,	$!(1 \multimap r)$.

Table 3.

After all these replacements we obtain a "good" sequent of the form

$$\Sigma_1, \Sigma_2, \Sigma_3 \Rightarrow$$

where Σ_1 is a multiset of the formulas of the kind from Table 3. Σ_2 is a multiset of new literals, and Σ_3 is a multiset of the formulas of the kind $!p$, where p is a new literal.

Then we transfer formulas from Σ_1 and Σ_3 to equivalent normal formulas. Namely, since $p \simeq (1 \multimap p)$, we may replace formulas $!p$ from Σ_3 by the Horn implication $!(1 \multimap p)$. Since $p^\bot \multimap q \simeq p\wp q$, we may replace $!(p^\bot \multimap q)$ by the simple disjunction $!(p\wp q)$. And using that $!((p \oplus q) \multimap r) \simeq {!(p \multimap r)} \otimes {!(q \multimap r)}$, we may replace the formulas of the form $!((p \oplus q) \multimap r)$ by the pair of the Horn implications $!(p \multimap r)$ and $!(q \multimap r)$. Finally, since $!(r \multimap p^\bot) \simeq (?(r \otimes p))^\bot$, we may remove the formulas $!(r \multimap p^\bot)$ from the antecedent, and add the formula $?(r \otimes p)$ to the consequent. After all, we obtain a sequent of the form:

$$\Sigma_2, !\Gamma \Rightarrow ?\Delta,$$

where Γ is a list of normal formulas and Δ is a list of simple products.

Then we replace the multiset of literals Σ_2 by simple product $W = \Sigma_2^\otimes$ and obtain a normal sequent $\Psi = (W, !\Gamma \Rightarrow ?\Delta)$. This sequent is derivable in LLW if and only if the sequent Φ is.

For LL this lemma is proved similarly except the point, where we use the fact that $\top \simeq 1$ in LLW. For LL it is proved that LL is reduced to the fragment without constants 0 and $\top$. ∎

Lemma 3.2 *Linear Logic and Linear Affine Logic are reduced (according to Turing) to the problems of the derivability of "good" sequents in the corresponding logic.*

Proof Note, that since $?A \simeq (!A^\bot)^\bot$ we can confine ourselves without loss of generality to the sequents without "?". Let we have any sequent Φ without "?". Consider the set $\mathcal{X} = \{A : \ !A \in \mathrm{Sub}\,\Phi\}$. With any formula A from $\mathcal{X}$ we associate a new literal p_A. For any formula $B \in \mathrm{Sub}\,\Phi$ we define the formula B' as a formula obtained from B by replacement of subformulas of the form $!A$ (which is not subformula of any other subformula of the kind $!C$) by the literal p_A. For example, if $B = (!q \multimap !(r \otimes !t))$ then $B' = (p_q \multimap p_{(r\otimes !t)})$. We define also the inverse operation ‘ as follows:

$$‘B = B[!A_1/p_{A_1}, \dots, !A_n/p_{A_n}],$$

where $\mathcal{X} = \{A_1, \dots, A_n\}$. It is clear that $‘(B') = B$.

Consider the multiset $\Pi_\mathcal{X}$ consisting of the following formulas:

$$(p_A \multimap p_A \otimes p_A),\ (p_A \multimap 1),\ (p_A \multimap A'),$$

for any $A \in \mathcal{X}$.

We shall also consider the multiset $\Sigma_\mathcal{X}$ with the following properties:

(a) For any set of the formulas $A_1, \dots, A_n, C \in \mathcal{X}$ (all A_i's are different) if the sequent

$$!\Pi_\mathcal{X}, !\Sigma_\mathcal{X}, p_{A_1}, p_{A_2}, \dots, p_{A_n} \Rightarrow C' \qquad (3)$$

is derivable, then

$$(p_{A_1} \otimes \dots \otimes p_{A_n} \multimap p_C) \in \Sigma_\mathcal{X}.$$

(b) $\Sigma_\mathcal{X}$ is the least set that meets the property (a), i.e. if any set $\tilde{\Sigma}$ meets the property (a) then $\Sigma_X \subseteq \tilde{\Sigma}$.

(It is easy to verify that the desired least set exists and finite. Moreover, if we have decidability algorithm for the "good" sequents of LL (LLW) then we can effectively construct this set, because the sequents of the form (3) are "good").

Lemma 3.2.1 *If $B \in \Pi_\mathcal{X}, \Sigma_X$, then the sequent $\Rightarrow$ $!‘B$ is derivable.*

Proof 1. If $B \in \Pi_X$, then $`B = (!A \multimap !A \otimes !A)$, or $(!A \multimap 1)$, or $(!A \multimap A)$. In all these cases the sequent $\Rightarrow `B$ is easily derivable, and so the sequent $\Rightarrow !`B$ is.

2. In order to prove this lemma for $B \in \Sigma_X$, we consider the set $\tilde{\Sigma}_{\mathcal{X}}$, which consists of the formulas of the form

$$B = (p_{A_1} \otimes \ldots \otimes p_{A_n} \multimap p_C)$$

(where $A_i, C \in \mathcal{X}$, and all A_i's are different), such that the sequent $\Rightarrow !`B$ is derivable. Let us verify that the set $\tilde{\Sigma}_{\mathcal{X}}$ meets the property (a). Indeed, suppose that the sequent

$$!\Pi_{\mathcal{X}}, !\tilde{\Sigma}_{\mathcal{X}}, p_{A_1}, \ldots, p_{A_n} \Rightarrow C'$$

is derivable, then we verify that $(p_{A_1} \otimes \ldots \otimes p_{A_n} \multimap C) \in \tilde{\Sigma}_{\mathcal{X}}$. If we substitute the formulas $!A$ for all literals p_A in the last sequent, then we get the derivable sequent

$$!`\Pi_{\mathcal{X}}, !`\tilde{\Sigma}_{\mathcal{X}}, !A_1, \ldots, !A_n \Rightarrow C. \tag{4}$$

For any formula $B \in \Pi_{\mathcal{X}}, \tilde{\Sigma}_{\mathcal{X}}$ the sequent $\Rightarrow !`B$ is derivable (item 1 of this lemma and the definition of $\tilde{\Sigma}_{\mathcal{X}}$). The sequent

$$!A_1, \ldots, !A_n \Rightarrow C$$

is derivable from these sequents and the sequent (4) by the rule (CUT). Thus, the sequent

$$\Rightarrow !(!A_1 \otimes \ldots \otimes !A_n \multimap !C)$$

is derivable. Hence, by the definition of the set $\tilde{\Sigma}_{\mathcal{X}}$,

$$(p_{A_1} \otimes \ldots \otimes p_{A_n} \multimap p_C) \in \tilde{\Sigma}_{\mathcal{X}}.$$

So, the set $\tilde{\Sigma}_{\mathcal{X}}$ meets the property (a). And therefore, by the property (b), $\Sigma_{\mathcal{X}} \subseteq \tilde{\Sigma}_{\mathcal{X}}$. Hence, if $B \in \Sigma_{\mathcal{X}}$ then the sequent $\Rightarrow !`B$ is derivable. ∎

Lemma 3.2.2 *If we have the sequent $\Phi = (\Gamma \Rightarrow \Delta)$ (which does not contain any "?"), then the sequent Φ is derivable if and only if the sequent*

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, \Gamma' \Rightarrow \Delta'$$

is derivable.

Proof Firstly, we prove that if the sequent

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, \Gamma' \Rightarrow \Delta'$$

is derivable then the sequent Φ is derivable too. If we substitute the formulas $!A$ for the literals p_A in this sequent, then we obtain the derivable sequent

$$!`\Pi_{\mathcal{X}}, !`\Sigma_{\mathcal{X}}, \Gamma \Rightarrow \Delta. \tag{5}$$

For any formula $B \in \Pi_{\mathcal{X}}, \Sigma_{\mathcal{X}}$ the sequent $\Rightarrow !`B$ is derivable (Lemma 3.2.1). The sequent $\Phi = (\Gamma \Rightarrow \Delta)$ is derivable from these sequents and the sequent (5) by the rule (CUT).

Now we prove by induction on the derivation of the sequent Φ that if this sequent is derivable, then the sequent

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, \Gamma' \Rightarrow \Delta'$$

is derivable too.

<u>Case 1.</u> Φ is derived by the rule $(L!)$:

$$\frac{A, \Gamma_0 \Rightarrow \Delta}{!A, \Gamma_0 \Rightarrow \Delta}\,(L!)$$

Then $A \in \mathcal{X}$. Hence, $(p_A \multimap A') \in \Pi_{\mathcal{X}}$. According to the induction hypothesis the sequent $!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, A', \Gamma_0' \Rightarrow \Delta'$ is derivable. Hence, the sequent $!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, \Gamma_0' \Rightarrow \Delta'$ is also derivable:

$$\cfrac{\cfrac{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, A', \Gamma_0' \Rightarrow \Delta' \qquad p_A, p_A \multimap A' \Rightarrow A'}{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, p_A \multimap A', \Gamma_0' \Rightarrow \Delta'}\,(CUT)}{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, \Gamma_0' \Rightarrow \Delta'}\,(L!, C!)$$

<u>Case 2.</u> Φ is derived by the rule $(W!)$:

$$\frac{\Gamma_0 \Rightarrow \Delta}{!A, \Gamma_0 \Rightarrow \Delta}\,(W!)$$

Then $A \in \mathcal{X}$. Hence, $(p_A \multimap 1) \in \Pi_{\mathcal{X}}$. By the induction hypothesis the sequent $!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, \Gamma_0' \Rightarrow \Delta'$ is derivable. Hence, the sequent $!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, \Gamma_0' \Rightarrow \Delta'$ is also derivable:

$$\cfrac{\cfrac{\cfrac{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, \Gamma_0' \Rightarrow \Delta'}{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, 1, \Gamma_0' \Rightarrow \Delta'}\,(L1) \qquad p_A, p_A \multimap 1 \Rightarrow 1}{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, p_A \multimap 1, \Gamma_0' \Rightarrow \Delta'}\,(CUT)}{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, \Gamma_0' \Rightarrow \Delta'}\,(L!, C!)$$

<u>Case 3.</u> Φ is derived by the rule $(C!)$:

$$\frac{!A, !A, \Gamma_0 \Rightarrow \Delta}{!A, \Gamma_0 \Rightarrow \Delta}\,(C!)$$

Then $A \in \mathcal{X}$. Hence, $(p_A \multimap p_A \otimes p_A) \in \Pi_{\mathcal{X}}$. By the induction hypothesis the sequent

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, p_A, \Gamma_0' \Rightarrow \Delta'$$

is derivable. The derivation of the sequent $!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, \Gamma_0' \Rightarrow \Delta'$ from this sequent is represented in Table 4.

<u>Case 4.</u> Φ is derived by the rule $(R!)$:

$$\frac{!A_1, \ldots, !A_n \Rightarrow C}{!A_1, \ldots, !A_n \Rightarrow !C}\,(R!)$$

$$\cfrac{\cfrac{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, p_A, \Gamma_0' \Rightarrow \Delta'}{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A \otimes p_A, \Gamma_0' \Rightarrow \Delta'} \qquad p_A, p_A \multimap p_A \otimes p_A \Rightarrow p_A \otimes p_A}{\cfrac{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, p_A \multimap p_A \otimes p_A, \Gamma_0' \Rightarrow \Delta'}{!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_A, \Gamma_0' \Rightarrow \Delta'}\,(L!, C!)}\,(CUT)$$

Table 4.

By the induction hypothesis the sequent

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_{A_1}, \ldots, p_{A_n} \Rightarrow C'$$

is derivable. Consider formulas $B_1, \ldots, B_m$, such that all B_i's are different and

$$\{B_1, \ldots, B_m\} = \{A_1, \ldots, A_n\}.$$

The sequent

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_{B_1}, \ldots, p_{B_m} \Rightarrow C'$$

is derivable (similarly to the case 3). Hence, by definition of the set $\Sigma_{\mathcal{X}}$,

$$(p_{B_1} \otimes \ldots \otimes p_{B_m} \multimap p_C) \in \Sigma_{\mathcal{X}}.$$

Therefore, the sequent

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_{B_1}, \ldots, p_{B_m} \Rightarrow p_C$$

is derivable. Similarly to the case 2, the sequent

$$!\Pi_{\mathcal{X}}, !\Sigma_{\mathcal{X}}, p_{A_1}, \ldots, p_{A_n} \Rightarrow p_C.$$

is derivable from the last sequent.

The other cases not associated with "!", are trivial.

■

It is easy to see that Lemma 3.2 follows from the last lemma. And Theorem 3 follows from Lemmas 3.1 and 3.2. ■

Theorems 1–3 provide the main

Theorem *Linear Affine Logic is decidable.*

5 Conclusion remarks

Here we consider one more corollary from the computational interpretation of the normal fragments. In [4] it was considered the certain fragment $H_2LL(!, \otimes, \multimap, \oplus)$. The sequents from this fragment were interpreted in terms of so-called non-determinate Petri Nets with conditional transitions. In fact, it is possible to express the rules of the game $\mathbf{A}_\Phi$ in the terms of these nets and vice versa. As a corollary of this fact we have that the whole Linear Logic is reduced to the fragment $H_2LL(!, \otimes, \multimap, \oplus)$. By analogous the multiplicative-exponential fragment of Linear Logic ($MELL = LL(!, ?, ^\perp, \multimap, \otimes, \wp, 1, \perp)$) is reduced to $H_2LL(!, \otimes, \multimap)$. Hence, the following fragments are decidable or undecidable simultaneously:

(i) $MELL = LL(!, ?, \otimes, \wp, \multimap, ^\perp, 1, \perp)$

(ii) $LL(!, \otimes, \multimap)$

(iii) The normal fragment of $LL(!, ?, \otimes, \wp, \multimap)$

The decidability problem of these fragments remains open.

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