Dependent Intersection: A New Way of Defining Records in Type Theory *

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Abstract

Records and dependent records are a powerful tool for programming, representing mathematical concepts, and program verification. In the last decade several type systems with records as primitive types were proposed. The question is arose: whether it is possible to define record type in existent type theories using standard types without introducing new primitives.

It was known that independent *records can be defined in type theories with dependent functions or intersection. On the other hand* dependent *records cannot be formed using standard types. Hickey introduced a complex notion of* very dependent functions *to represent dependent records. In the current paper we extend Martin-Löf's type theory with a simpler type constructor* dependent intersection*, i.e., the intersection of* two *types, where the second type may depend on elements of the first one (not to be confused with the intersection of a family of types). This new type constructor allows us to define dependent records in a very simple way. It also allows us to define the set type constructor.*

1 Introduction

1.1 Type Theory

We will use the NuPRL type theory [6], which is an extension of Martin-Löf's type theory [16]. Martin-Löf's type theory allows dependent types. That is, type expression may contain free variables ranging over arbitrary types. For example, we can form an expression $T[x] = [0..x]$ which represents an initial sequent of natural numbers. This expression is a type when $x \in \mathbb{N}$. (Some notations: we will use $T[x_1, \ldots, x_n]$ for expressions that may contain free variables x_1, …, x_n (and probably some other free variables), and $T[t_1, \ldots, t_n]$ for substitution terms t_i's for *all* free occurrences of x_i's).

Martin-Löf's type theory has the following judgments:

$A\ \mathtt{Type}$	A is a well-formed type
$A = B$	A and B are (intentionally) equal types
$a \in A$	a has type A
$a = b \in A$	a and b are equal as elements of type A

The NuPRL type theory also has subtyping relation. Although it is not essential for our work, we should mention that membership and subtyping in NuPRL are extensional. For example, $A \subseteq B$ does not say anything about structure of these types, but only means that if $x \in A$ then $x \in B$. As a result the type checking and subtyping are undecidable. On the other hand, type equality ($A = B$) is intensional. We will use $A =_e B$ for extensional equality: $A =_e B \triangleq (A \subseteq B)\&(B \subseteq A)$.

The NuPRL type theory has also an intersection type. The intersection of two types A and B is a new type containing elements that are both in A and B. For example, $\lambda x.x + 1$ is an element of the type $(\mathbb{Z} \to \mathbb{Z}) \cap (\mathbb{N} \to \mathbb{N})$. Two elements are considered to be equal as elements of the type $A \cap B$ if they are equal in both types A and B.

Example 1 *Let $A = \mathbb{N} \to \mathbb{N}$ and $B = \mathbb{Z}^- \to \mathbb{Z}$ (where $\mathbb{Z}^-$ is a type of negative integers). Let id be $\lambda x.x$ and abs be $\lambda x.|x|$. Then id and abs are both elements of the type $A \cap B$. Although id and abs are equal as elements of the type $\mathbb{N} \to \mathbb{N}$ (because these two functions do not differ on $\mathbb{N}$), id and abs are different as elements of $\mathbb{Z}^- \to \mathbb{Z}$. Therefore, $id \neq abs \in A \cap B$.*

In Martin-Löf's type theory types are first-class objects. There is the universe type $\mathbb{U}$ that contain types that were formed without using of $\mathbb{U}$.

Our work is implemented in a setting of the NuPRL type theory, namely in the MetaPRL system [12, 13] . See theories `itt_disect` and `itt_record` in Logical Theories

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in [13]. All proofs except the proof of the semantical Theorem 10 are machine-checked. We believe that most of our results could be adapted to any type theory that allows binary intersection and dependent types.

1.2 Records

In general, records are tuples of labeled fields, where each field may have its own type. In dependent records (or more formally dependently typed records) the type of components may depend on values of the other components. Since we have the type of types $\mathbb{U}$, values of record components may be types. This makes the notion of dependent records very powerful. Dependent records may be used to represent algebraic structures (such as groups) and modules in programming languages like SML or Haskell (see for example [3, 10]).

Example 2 *One can define the signature for ordered set as a dependent record type:*

$$OrdSetSig = \{\mathtt{t} : \mathbb{U}; \mathtt{less} : \mathtt{t} \rightarrow \mathtt{t} \rightarrow \mathrm{Bool}\}$$

This definition can be understood as an algebraic structure as well as an interface of a module in a programing language.

Example 3 *The proposition-as-type principle allows us to add the property of ordered sets as a new component:*

$$OrdSet = \{\mathtt{t}:\mathbb{U}; \mathtt{less}:\mathtt{t} \rightarrow \mathtt{t} \rightarrow \mathrm{Bool}; \mathtt{axm} : Ord(\mathtt{t}, \mathtt{less})\}$$

where $Ord(\mathtt{t}, \mathtt{less})$ is a predicate stating that `less` *is a transitive irreflexive relation on* `t`. *Here* `axm` *is a new field that defines the axiom of the algebraic structure of ordered sets (or specification of the module type $OrdSet$).*

Example 4 *In type theories with equality, manifested fields ([15]) may be also represented as specification.*

$$IntOrdSetSig = \{\mathtt{t}:\mathbb{U}; \mathtt{less}:\mathtt{t}\rightarrow\mathtt{t}\rightarrow\mathrm{Bool}; \mathtt{mnf}:\mathtt{t}=\mathbb{Z}\}$$

is a signature where `t` *is bound to be the type of integers.*

From a mathematical point of view the record type is similar to the product type. The essential difference is the subtyping property: we can extend a record type with new fields and get a subtype of the original record type. E.g. $OrdSet$ and $IntOrdSetSig$ defined above are subtypes of $OrdSetSig$. The subtyping property is important in mathematics: we can apply all theorems about monoid's to smaller types such as groups. It is also essential in programing for inheritance and abstractions.

Different type theories with records were proposed both for proof systems as well as for programming languages ([10, 15, 9, 3, 4, 19] and others). These systems treat the record type as a new primitive. In the current paper we are interesting in the following natural question: *is it possible to express the notion of records in usual type theories without record type as primitives?* This question is especially interesting for pure mathematical proof systems. As we saw records are a handy tool to represent algebraic structures. On the other hand records do not seem to be the basic mathematical concept that should be included in the foundation of mathematics. Records should be rather defined in terms of more abstract mathematical concepts.

It is known that it is possible to define *independent records* in a sufficient powerful type theory that has dependent functions [11] or intersection [21]. On the other hand, there is no known way to form dependent records in standard Martin-Löf's type theory [4]. However, Hickey [11] showed that *dependent records* can be formed in an extension of Martin-Löf's type theory. Namely, he introduced a new type of *very dependent functions*. This type is powerful enough to express dependent records in a type theory and provides a mathematical foundation of dependent records. Unfortunately the type of very dependent functions is very complex itself. The rules and the semantics probably is more complicated for this type than for dependent records. The question is whether there is a simpler way to add dependent records to a type theory.

In this paper we extend the NuPRL type theory with a simpler and easier to understand primitive type constructor, *dependent intersection*. This is a natural generalization of the standard intersection introduced in [8] and [20]. Dependent intersection is an intersection of *two* types, where the second type may depend on elements of the first one. This type constructor is built by analogy to dependent products: elements of dependent product are pairs where the type of the second component may depend on the first component. We will show that dependent intersection allows us to define the record type in a very simple way. Our definition of records is extensionally equal to Hickey's ones, but is far simpler. Moreover our constructors (unlike Hickey's) allow us to extend record types. For example, having a definition of monoids we can define groups by extending this definition rather than repeating the definition of monoid.

1.3 The Set Type Constructor

The NuPRL type theory has a primitive type constructor for subset types. By definition, the set type $\{x : T \mid P[x]\}$ is a subtype of T, which contains only such elements x of T that satisfy property $P[x]$ (see [6]).

Example 5 *The type of natural numbers is defined as $\mathbb{N} = \{n : \mathbb{Z} \mid n \geq 0\}$. Without set types we would have to define $\mathbb{N}$ as $n : \mathbb{Z} \times (n \geq 0)$. In this case we would not have the subtyping property $\mathbb{N} \subseteq \mathbb{Z}$.*

It turns out that dependent intersection can be also used to define a set type. This means that dependent intersection not only adds support for dependent records, it *simplifies* the overall the NuPRL type theory at the same time.

2 Dependent Intersection

We extend the definition of intersection $A \cap B$ to a case when type B can depend on elements of type A. Let A be a type and $B[x]$ be a type for all x of type A. We define a new type, *dependent intersection* $x{:}A \cap B[x]$. This type contains all elements a from A such that a is also in $B[a]$.

Remark 6 *Do not confuse* the dependent intersection *with* the intersection of a family of types $\bigcap_{x:A} B[x]$. *The latter refers to an intersection of types $B[x]$ for all x in A. The difference between these two type constructors is similar to the difference between dependent products $x{:}A \times B[x] = \Sigma_{x:A} B[x]$ and the product of a family of types $\Pi_{x:A} B[x] = x : A \to B[x]$.*

Example 7 *The ordinary binary intersection is just a special case of a dependent intersection with a constant second argument: $A \cap B = x : A \cap B$.*

Example 8 *Let $A = \mathbb{Z}$ and $B[x] = [0 .. x^2{-}5]$. Then $x : A \cap B[x]$ is a set of all integers, such that $0 \leq x \leq x^2{-}5$.*

Two elements a and a' are equal in the dependent intersection $x{:}A \cap B[x]$ when they are equal both in A and $B[a]$.

Example 9 *Let A be $\{0\} \to \mathbb{N}$ and $B[f]$ be $\{1\} \to [0 .. f(0)]$, where $\{0\}$ and $\{1\}$ are types that contain only one element (0 and 1 respectively). Then $x{:}A \cap B[x]$ is a type of functions f that map 0 to a natural number n_0 and map 1 to a natural number $n_1 \in [0 .. n_0]$. Two such functions f and f' are equal in this type, when first, $f = f' \in \{0\} \to \mathbb{N}$, i.e. $f(0) = f'(0)$, and second, $f = f' \in \{1\} \to [0 .. f(0)]$, i.e. $f(1) = f'(1) \leq f(0)$.*

2.1 Semantics

We are going to give the formal semantics for dependent intersection types based on the predicative PER semantics for the NuPRL type theory [1, 2]. In the PER semantics types are interpreted as partial equivalence relations (PERs) over terms. Partial equivalence relations are relations that transitive and symmetric, but not necessary reflexive.

According to [2], to give the semantics for a type expression A we need to determine when this expression is a well-formed type, define elements of this type, and specify the partial equivalence relation on terms for this type ($a = b \in A$). We should also give an equivalence relation on types, i.e. determine when two types are equal. See [2] for details.

$$\frac{\Gamma \vdash A\;\texttt{Type} \qquad \Gamma; x : A \vdash B[x]\;\texttt{Type}}{\Gamma \vdash (x : A \cap B[x])\;\texttt{Type}}$$

$$\frac{\Gamma \vdash A = A' \qquad \Gamma; x : A \vdash B[x] = B'[x]}{\Gamma \vdash (x : A \cap B[x]) = (x : A' \cap B'[x])}$$

$$\frac{\Gamma \vdash a \in A \qquad \Gamma \vdash a \in B[a] \qquad \Gamma \vdash x : A \cap B[x]\;\texttt{Type}}{\Gamma \vdash a \in (x : A \cap B[x])}$$

$$\frac{\Gamma \vdash a{=}a'{\in}A \quad \Gamma \vdash a{=}a'{\in}B[a] \quad \Gamma \vdash x{:}A \cap B[x]\;\texttt{Type}}{\Gamma \vdash a = a' \in (x : A \cap B[x])}$$

$$\frac{\Gamma; u : (x : A \cap B[x]); \Delta; x : A; y : B[x] \vdash C[x, y]}{\Gamma; u : (x : A \cap B[x]); \Delta \vdash C[u, u]}$$

Table 1. Rules for dependent intersection

The Extension of the Semantics We introduce a new term constructor for dependent intersection $x : A \cap B[x]$. This constructor bounds the variable x in $B[x]$. We extend the semantics of [2] as follows.

- The expression $x : A \cap B[x]$ is a well-formed type if and only if A is a type and $B[x]$ is a functional type over $x : A$. That is, for any x from A the expression $B[x]$ should be a type and if $x = x' \in A$ then $B[x] = B[x']$.
- The elements of the well-formed type $x : A \cap B[x]$ are such terms a that a is an element of both types A and $B[a]$.
- Two elements a and a' are equal in the well-formed type $x : A \cap B[x]$ iff $a = a' \in A$ and $a = a' \in B[a]$.
- Two types $x : A \cap B[x]$ and $x : A' \cap B'[x]$ are equal when A and A' are equal types and for all x and y from A if $x = y \in A$ then $B[x] = B'[y]$.

2.2 The Inference Rules

The corresponding inference rules are shown in Table 1.

Theorem 10 *All rules of Table 1 are valid in the semantics given above.*

This theorem is proved by straightforward application of the semantics definition.

Theorem 11 *The following rules can be derived from the primitive rules of Table 1 in a type theory with the appropriate cut rule.*

$$\frac{\Gamma \vdash a = a' \in (x : A \cap B[x])}{\Gamma \vdash a = a' \in A}$$

$$\frac{\Gamma \vdash a = a' \in (x : A \cap B[x])}{\Gamma \vdash a = a' \in B[a]}$$

Theorem 12 *Dependent intersection is associative, i.e.*

$$x : A \cap (y : B[x] \cap C[x, y]) =_e z : (x : A \cap B[x]) \cap C[z, z]$$

The formal proof is checked by the MetaPRL system. We show here a sketch of a proof. An element x has type $a : A \cap (b : B[a] \cap C[a, b])$ iff it has types A and $b : B[x] \cap C[x, b]$. The latter is a case iff $x \in B[x]$ and $x \in C[x, x]$. On the other hand, x has type $ab : (a : A \cap B[a]) \cap C[ab, ab]$ iff $x \in (a : A \cap B[a])$ and $x \in C[x, x]$. The former means that $x \in A$ and $x \in B[x]$. Therefore $x \in a : A \cap (b : B[a] \cap C[a, b])$ iff $x \in A$ and $x \in B[x]$ and $x \in C[x, x]$ iff $x \in ab : (a : A \cap B[a]) \cap C[ab, ab]$.

3 Records

We are going to define record types using dependent intersection. In this section we informally describe what properties we are expecting from records. The formal definitions are presented in Section 4.

3.1 Plain Records

Records are collection of labeled fields. We use the following notations for records:

$$\{\mathtt{x}_1 = a_1; \ldots; \mathtt{x}_n = a_n\} \qquad (1)$$

where $\mathtt{x}_1, \ldots, \mathtt{x}_n$ are *labels* and $a_1, \ldots a_n$ are corresponding fields. Usually labels have a string type, but generally speaking labels can be of any fixed type $Label$ with a decidable equality. We will use the `true type` font for labels.

The selection operator $r.\mathtt{x}$ is used to access record fields. If r is a record then $r.\mathtt{x}$ is a field of this record labeled $\mathtt{x}$. That is we expect the following reduction rule:

$$\{\mathtt{x}_1 = a_1; \ldots; \mathtt{x}_n = a_n\}.\mathtt{x}_i \longrightarrow a_i.$$

Fields may have different types. If each a_i has type A_i then the whole record (1) has the type

$$\{\mathtt{x}_1 : A_1; \ldots; \mathtt{x}_n : A_n\}. \qquad (2)$$

Also we want the natural typing rule for the field selection: for any record r of the type (2) we should be able to conclude that $r.\mathtt{x}_i \in A_i$.

The main difference between record types and products $A_1 \times \cdots \times A_n$ is the that record type has the *subtyping property*. Given two records R_1 and R_2, if any label declared in R_1 as a field of type A is also declared in R_2 as a field of type B, such that $B \subseteq A$, then R_2 is subtype of R_1. In particular,

$$\{\mathtt{x}_1 : A_1; \ldots; \mathtt{x}_n : A_n\} \subseteq \{\mathtt{x}_1 : A_1; \ldots; \mathtt{x}_m : A_m\} \qquad (3)$$

where $m < n$.

Example 13 *Let* $Point = \{\mathtt{x} : \mathbb{Z}; \mathtt{y} : \mathbb{Z}\}$ *and* $ColorPoint = \{\mathtt{x} : \mathbb{Z}; \mathtt{y} : \mathbb{Z}; \mathtt{color} : Color\}$. *Then the record* $\{\mathtt{x} = 0; \mathtt{y} = 0; \mathtt{color} = red\}$ *is not only a* $ColorPoint$*, but it is also a* $Point$*, so we can use this record whenever* $Point$ *is expected. For example, we can use it as an argument of the function of the type* $Point \to T$*. Further the result of this function does not depend whether we use* $\{\mathtt{x} = 0; \mathtt{y} = 0; \mathtt{color} = red\}$ *or* $\{\mathtt{x} = 0; \mathtt{y} = 0; \mathtt{color} = green\}$*. That is, these two records are equal as elements of the type* $Point$*, i.e.*

$$\{\mathtt{x} = 0; \mathtt{y} = 0; \mathtt{color} = red\} = \\ \{\mathtt{x} = 0; \mathtt{y} = 0; \mathtt{color} = green\} \in \{\mathtt{x} : \mathbb{Z}; \mathtt{y} : \mathbb{Z}\}$$

Using subtyping one can model the private fields. Consider a record r that has one "private" field $\mathtt{x}$ of the type A and one "public" field $\mathtt{y}$ of the type B. This record has the type $\{\mathtt{x} : A; \mathtt{y} : B\}$ Using subtyping property we can conclude that it also has type $\{\mathtt{y} : B\}$. Now we can consider type $\{\mathtt{y} : B\}$ as a public interface for this record. A user knows only that $r \in \{\mathtt{y} : B\}$. Therefore he has access to field $\mathtt{y}$, but access to field $\mathtt{x}$ would be type invalid (i.e. untyped). Formally it meant that a function of the type $\{\mathtt{y} : B\} \to T$ can assess only the field $\mathtt{y}$ on its argument (although an argument of this function can have other fields).

Further, records do not depend on field ordering. For example, $\{\mathtt{x} = 0; \mathtt{y} = 1\}$ should be equal to $\{\mathtt{y} = 1; \mathtt{x} = 0\}$, moreover $\{\mathtt{x} : A; \mathtt{y} : B\}$ and $\{\mathtt{y} : B; \mathtt{x} : A\}$ should define the same type.

3.1.1 Records as Dependent Functions

Records may be considered as mappings from labels to the corresponding fields. Therefore it is natural to define a record type as a function type with the domain $Label$ (cf. [5]). Since the types of each field may vary, one should use dependent function type (i.e., Π type). Let $Field[l]$ be a type of a field labeled l. For example, for the record type (2) take

$$\begin{aligned} Field[l] \triangleq\ & \mathtt{if}\ l = \mathtt{x}_1\ \mathtt{then}\ A_1\ \mathtt{else} \\ & \ldots \\ & \mathtt{if}\ l = \mathtt{x}_n\ \mathtt{then}\ A_n\ \mathtt{else}\ \mathrm{Top} \end{aligned}$$

Then define the record type as the dependent function type:[1]

$$\{\mathtt{x}_1 : A_1; \ldots; \mathtt{x}_n : A_n\} \triangleq l : Label \to Field[l]. \quad (4)$$

Now records may be defined as functions:

$$\begin{aligned} &\{\mathtt{x}_1 = a_1; \ldots; \mathtt{x}_n = a_n\} \triangleq \\ &\quad \lambda l.\mathtt{if}\ l = \mathtt{x}_1\ \mathtt{then}\ a_1\ \mathtt{else} \\ &\quad\quad \ldots \\ &\quad\quad \mathtt{if}\ l = \mathtt{x}_n\ \mathtt{then}\ a_n \end{aligned} \quad (5)$$

And selection is defined as application:

$$r.l \triangleq r\ l \quad (6)$$

One can see that these definitions meet the expecting properties mentioned above including subtyping property.

3.1.2 Records as Intersections

Using the above definitions we can prove that in case when all $\mathtt{x}_i$'s are distinct labels

$$\{\mathtt{x}_1 : A_1; \ldots; \mathtt{x}_n : A_n\} =_e \{\mathtt{x}_1 : A_1\} \cap \ldots \cap \{\mathtt{x}_n : A_n\}. \quad (7)$$

This property provides us a simpler way to define records. First, let us define the type of records with only one field. We define it as a function type like we did it in the last section, but for single-field records we do not need dependent functions, so we may simplify the definition:

$$\{\mathtt{x} : A\} \triangleq \{\mathtt{x}\} \to A \quad (8)$$

where $\{\mathtt{x}\}$ is the singleton subset of type $Label$. Now we may take (7) and (8) as a definition of an arbitrary record type instead of (4) and keep definitions (5) and (6). This way was used in [21] where $\{\mathtt{x} : A\}$ was a primitive type.

Example 14 *The record $\{\mathtt{x} = 1; \mathtt{y} = 2\}$ by definition (5) is a function that maps $\mathtt{x}$ to 1 and $\mathtt{y}$ to 2. Therefore it has type $\{\mathtt{x}\} \to \mathbb{Z} = \{\mathtt{x} : \mathbb{Z}\}$ and also has type $\{\mathtt{y}\} \to \mathbb{Z} = \{\mathtt{y} : \mathbb{Z}\}$. Hence it has type $\{\mathtt{x} : \mathbb{Z}; \mathtt{y} : \mathbb{Z}\} = \{\mathtt{x} : \mathbb{Z}\} \cap \{\mathtt{y} : \mathbb{Z}\}$.*

One can see that when all labels are distinct definitions (4) and (7)+(8) are equivalent. That is, for any record expression $\{x_1 : A_1; \ldots; x_n : A_n\}$ where $x_i \neq x_j$, these two definitions define two extensionally equal types.

However, definitions (7)+(8) differ from the traditional ones, in the case when labels coincide. Most record calculi prohibit repeating labels in the declaration of record types, e.g., they do not recognize the expression $\{\mathtt{x} : A; \mathtt{x} : B\}$ as a valid type. On the other hand, in [11] in the case when labels coincide the last field overlap the previous ones, e.g., $\{\mathtt{x} : A; \mathtt{x} : B\}$ is equal to $\{\mathtt{x} : B\}$. In both these cases many typing rules of the record calculus need some additional conditions that prohibits coincident labels. For example, the subtyping relation (3) would be true only when all labels $\mathtt{x}_i$ are distinct.

We will follow the definition (7) and allow repeated labels and assume that

$$\{\mathtt{x} : A; \mathtt{x} : B\} = \{\mathtt{x} : A \cap B\}. \quad (9)$$

This may look unusual, but this notation significantly simplifies the rules of the record calculus, because we do not need to worry about coincident labels. Moreover, this allow us to have multiply inheriting (see Section 4.3.2 for an example). Note that the equation (9) holds also in [7].

3.2 Dependent Records

We want to be able to represent abstract data types and algebraic structures as records. For example, a semigroup may be considered as a record with the fields $\mathtt{car}$ (representing a carrier) and $\mathtt{product}$ (representing a binary operation). The type of $\mathtt{car}$ is the universe $\mathbb{U}$. The type of $\mathtt{product}$ should be $\mathtt{car} \times \mathtt{car} \to \mathtt{car}$. The problem is that the type of $\mathtt{product}$ depends on the value of the field $\mathtt{car}$. Therefore we cannot use plain record types to represent such structures.

We need dependent records [4, 11, 19]. In general a dependent record type has the following form

$$\{\mathtt{x} : A; \mathtt{y} : B[\mathtt{x}]; \mathtt{z} : C[\mathtt{x}, \mathtt{y}]; \ldots\} \quad (10)$$

That is, the type of a field in such records can depend on the values of the previous fields.

The following main property show the intended meaning of this type.

> The record $\{\mathtt{x} = a; \mathtt{y} = b; \mathtt{z} = c; \ldots\}$ has type (10) if and only if
>
> $$a \in A, \quad b \in B[a], \quad c \in C[a, b], \quad \ldots$$

Example 15 *Let $SemigroupSig$ be the record type that represents the signature of semigroups:*

$$SemigroupSig \triangleq \{\mathtt{car} : \mathbb{U}; \mathtt{product} : \mathtt{car} \times \mathtt{car} \to \mathtt{car}\}.$$

Semigroups are elements of $SemigroupSig$ satisfying the associative axiom. This axiom may be represented as an additional field:

$$\begin{aligned} Semigroup \triangleq \{&\mathtt{car} : \mathbb{U}; \\ &\mathtt{product} : \mathtt{car} \times \mathtt{car} \to \mathtt{car}; \\ &\mathtt{axm} : \forall x, y, z : \mathtt{car}.\ (x \cdot y) \cdot z = x \cdot (y \cdot z)\} \end{aligned}$$

where $x \cdot y$ stands for $\mathtt{product}(x, y)$.

[1] We use the standard NuPRL's notations $x : A \to B[x] = \prod_{x:A} B[x]$ for the type of functions that maps each $x \in A$ to an element of the type $B[x]$.

3.2.1 Dependent Records as Very Dependent Functions

We cannot define dependent record type using the ordinary dependent function type, because the type of the fields depends not only on labels, but also on values of other fields.

To represent dependent records Hickey [11] introduced the *very dependent function* type constructor:

$$\{f \mid x : A \to B[f, x]\} \tag{11}$$

Here A is the domain of the function type and the range $B[f, x]$ can depend on the argument x and the function f itself. That is, type (11) refers to the type of all functions g with the domain A and the range $B[g, a]$ on any argument $a \in A$.

For instance, $SemigroupSig$ can be represented as a very dependent function type

$$SemigroupSig \triangleq \{r \mid l : Label \to Field[r, l]\} \tag{12}$$

where $Field[r, l] \triangleq$

$$\begin{array}{l} \texttt{if } l = \texttt{car then } \mathbb{U} \texttt{ else} \\ \texttt{if } l = \texttt{product then } r.\texttt{car} \times r.\texttt{car} \to r.\texttt{car} \\ \quad \texttt{else } \mathrm{Top} \end{array}$$

Not every very dependent function type has a meaning. For example the range of the function on argument a cannot depend on $f(a)$ itself. For instance, the expression

$$\{f \mid x : A \to f(x)\}$$

is not a well-formed type.

The type (11) is well-formed if there is some well-founded order $<$ on the domain A, and the range type $B[x, f]$ on $x = a$ depends only on values $f(b)$, where $b < a$. The requirement of well-founded order makes the definition of very-dependent functions to be very complex. See [11] for more details.

3.2.2 Dependent Records as Dependent Intersection

By using dependent intersection we can avoid the complex concept of very dependent functions. For example, we may define

$$\begin{array}{l} SemigroupSig \triangleq \mathit{self} : \{\texttt{car} : \mathbb{U}\} \cap \\ \{\texttt{product} : \mathit{self}.\texttt{car} \times \mathit{self}.\texttt{car} \to \mathit{self}.\texttt{car}\} \end{array}$$

Here self is a bound variable that is used to refer to the record itself considered as a record of the type $\{\texttt{car} : \mathbb{U}\}$. This definition can be read as following:

> r has type $SemigroupSig$, when first, r is a record with a field car of the type $\mathbb{U}$, and second, r is a record with a field product of the type $r.\texttt{car} \times r.\texttt{car} \to r.\texttt{car}$.

This definition of the $SemigroupSig$ type is extensionally equal to (12), but it has two advantages. First, it is much simpler. Second, dependent intersection allows us to extend the $SemigroupSig$ type to the $Semigroup$ type by adding an extra field axm:

$$\begin{array}{l} Semigroup \triangleq \mathit{self} : SemigroupSig \cap \\ \{\texttt{axm} : \forall x, y, z : \mathit{self}.\texttt{car} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)\} \end{array}$$

where $x \cdot y$ stands for $\mathit{self}.\texttt{product}(x, y)$.

We can define a dependent record type of an arbitrary length in this fashion as a dependent intersection of single-field records associated to the left.

Note that $Semigroup$ can be also defined as an intersection associated to the right: $Semigroup =$

$$\begin{array}{rl} r_c : & \{\texttt{car} : \mathbb{U}\} \cap \\ \big(r_p : & \{\texttt{product} : r_c.\texttt{car} \times r_c.\texttt{car} \to r_c.\texttt{car}\} \cap \\ & \{\texttt{axm} : \forall x, y, z : r_c.\texttt{car} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)\}\big) \end{array}$$

where $x \cdot y$ stands for $r_p.\texttt{product}(x, y)$. Here r_c and r_p are bound variables. Both of them refer to the record itself, but r_c has type $\{\texttt{car} : \mathbb{U}\}$ and r_p has type $\{\texttt{product} : \ldots\}$. These two definitions are equal, because of associativity of dependent intersection (Theorem 12).

Note that Pollack [19] considered two types of dependent records: left associating records and right associating records. However, in our framework left and right association are just two different ways of building the same type. We will allow using both of them. Which one to chose is the matter of taste.

4 The Record Calculus

4.1 The Formal Definitions

Now we are going to give the formal definitions of records using dependent intersection.

4.1.1 Records

Elements of record types are defined as functions from labels to the corresponding fields. We need three primitive operations:

1. Empty record: $\{\} \triangleq \lambda l.l$
   (We could pick any function as a definition of an empty record.)
2. Field update/extension:
   $$r.(\mathtt{x} := a) \triangleq (\lambda l.\texttt{if } l = \mathtt{x} \texttt{ then } a \texttt{ else } r\ l)$$
3. Field selection: $r.\mathtt{x} \triangleq r\ x$

Reduction rules

$$(r.\mathtt{x} := a).\mathtt{x} \longrightarrow a \qquad\qquad (r.\mathtt{y} := b).\mathtt{x} \longrightarrow r.\mathtt{x} \text{ when } \mathtt{x} \neq \mathtt{y}$$

Single-field record

$$\frac{\Gamma \vdash A\ \mathtt{Type} \quad \Gamma \vdash \mathtt{x} \in \mathit{Label}}{\Gamma \vdash \{\mathtt{x} : A\}\ \mathtt{Type}} \qquad \frac{\Gamma \vdash a \in A \quad \Gamma \vdash \mathtt{x} \in \mathit{Label}}{\Gamma \vdash r.\mathtt{x} := a \in \{\mathtt{x} : A\}} \qquad \frac{\Gamma \vdash r \in \{\mathtt{x} : A\} \quad \Gamma \vdash \mathtt{x} \neq \mathtt{y} \in \mathit{Label}}{\Gamma \vdash (r.\mathtt{y} := b) = r \in \{\mathtt{x} : A\}} \qquad \frac{\Gamma \vdash r \in \{\mathtt{x} : A\}}{\Gamma \vdash r.\mathtt{x} \in A}$$

Independent record

$$\frac{\Gamma \vdash R_1\ \mathtt{Type} \quad \Gamma \vdash R_2\ \mathtt{Type}}{\Gamma \vdash \{R_1; R_2\}\ \mathtt{Type}} \qquad \frac{\Gamma \vdash r \in R_1 \qquad \Gamma \vdash r \in R_2}{\Gamma \vdash r \in \{R_1; R_2\}} \qquad \frac{\Gamma \vdash r \in \{R_1; R_2\}}{\Gamma \vdash r \in R_1 \qquad \Gamma \vdash r \in R_2}$$

Left associating record

$$\frac{\Gamma \vdash R_1\ \mathtt{Type} \qquad \Gamma; \mathit{self} : R_1 \vdash R_2[\mathit{self}]\ \mathtt{Type}}{\Gamma \vdash \{R_1; R_2[\mathit{self}]\}\ \mathtt{Type}}$$

$$\frac{\Gamma \vdash r \in R_1 \quad \Gamma \vdash r \in R_2[r] \quad \Gamma \vdash \{R_1; R_2[\mathit{self}]\}\ \mathtt{Type}}{\Gamma \vdash r \in \{R_1; R_2[\mathit{self}]\}}$$

$$\frac{\Gamma \vdash r \in \{R_1; R_2[\mathit{self}]\}}{\Gamma \vdash r \in R_1 \qquad \Gamma \vdash r \in R_2[r]}$$

Right associating record

$$\frac{\Gamma \vdash \{\mathtt{x} : A\}\ \mathtt{Type} \qquad \Gamma; x : A \vdash R[x]\ \mathtt{Type}}{\Gamma \vdash \{x : \mathtt{x} : A; R[x]\}\ \mathtt{Type}}$$

$$\frac{\Gamma \vdash r \in \{\mathtt{x}{:}A\} \quad \Gamma \vdash r \in R[r.\mathtt{x}] \quad \Gamma \vdash \{x{:}\mathtt{x}{:}A; R[x]\}\ \mathtt{Type}}{\Gamma \vdash r \in \{x{:}\mathtt{x}{:}A; R[x]\}}$$

$$\frac{\Gamma \vdash r \in \{x : \mathtt{x} : A; R[x]\}}{\Gamma \vdash r.\mathtt{x} \in A \qquad \Gamma \vdash r \in R[r.\mathtt{x}]}$$

Table 2. Inference rules for records

We can construct any record by these operations: we define $\{\mathtt{x}_1 = a_1; \ldots; \mathtt{x}_n = a_n\}$ as

$$\{\}.(\mathtt{x}_1 := a_1).(\mathtt{x}_2 := a_2). \ \ldots \ .(\mathtt{x}_n := a_n)$$

4.1.2 Record Types

Single-field record type is defined as

$$\{\mathtt{x} : A\} \triangleq \{\mathtt{x}\} \to A$$

where $\{\mathtt{x}\} \triangleq \{l : \mathit{Label} \mid l = \mathtt{x} \in \mathit{Label}\}$ is a singleton set.

Independent concatenation of record types is defined as

$$\{R_1; R_2\} \triangleq R_1 \cap R_2$$

This definition is a partial case of the bellow definition of left associating records when R_2 does not depend on *self*.

Left associating dependent concatenation of record types is defined as

$$\{\mathit{self} : R_1; R_2[\mathit{self}]\} \triangleq \mathit{self} : R_1 \cap R_2[\mathit{self}]$$

Syntactical Remarks Here variable *self* is bounded in R_2. When we use the name "self" for this variable, we can use the shortening $\{R_1; R_2[\mathit{self}]\}$ for this type. Further, we will omit "*self*." in the body of R_2, e.g. we will write just $\mathtt{x}$ for $\mathit{self}.\mathtt{x}$, when such notation does not lead to misunderstanding. We assume that this concatenation is a left associative operation and we will omit inner braces. For example, we will write $\{\mathtt{x} : A;\mathtt{y} : B[\mathit{self}];\mathtt{z} : C[\mathit{self}]\}$ instead of $\{\{\{\mathtt{x} : A\}; \{\mathtt{y} : B[\mathit{self}]\}\}; \{\mathtt{z} : C[\mathit{self}]\}\}$. Note that in this expression there are two distinct bound variable *self*. First one is bound in B and refers to the record itself as a record of the type $\{\mathtt{x} : A\}$. Second *self* is bound in C, it also refers to the same record, but it has type $\{\mathtt{x} : A; \mathtt{y} : B[\mathit{self}]\}$.

Right associating dependent concatenation. The above definitions are enough to form any record type, but to complete the picture we give the definition of right associating record constructor:

$$\{x : \mathtt{x} : A; R[x]\} \triangleq \mathit{self} : \{\mathtt{x} : A\} \cap R[\mathit{self}.\mathtt{x}]$$

Syntactical Remarks Here x is a variable bound in R that represents a field $\mathtt{x}$. Note that we may α-convert the variable x, but not a label $\mathtt{x}$, e.g., $\{x : \mathtt{x} : A; R[x]\} = \{y : \mathtt{x} : A; R[y]\}$, but $\{x : \mathtt{x} : A; R[x]\} \neq \{y : \mathtt{y} : A; R[y]\}$. We will usually use the same name for labels and corresponding bound variables. This connection is right associative, e.g., $\{x : \mathtt{x} : A; y : \mathtt{y} : B[x]; \mathtt{z} : C[x, y]\}$ stands for $\{x : \mathtt{x} : A; \{y : \mathtt{y} : B[x]; \{\mathtt{z} : C[x, y]\}\}\}$.

4.2 The Rules

The basic rules of our record calculus are shown in Table 2.

Theorem 16 *All the rules of Table 2 are derivable from the definitions given above.*

From the reduction rules we get:

$$\{\mathtt{x}_1 = a_1; \ldots; \mathtt{x}_n = a_n\}.\mathtt{x}_i \longrightarrow a_i$$

when all $\mathtt{x}_i$'s are distinct.

We do not show the equality rules here, because in fact, these rules repeat rules in Table 2 and can be derived from them using substitution rules in the NuPRL type theory. For example, we can prove the following rules

$$\frac{\Gamma \vdash a = a' \in A \qquad \Gamma \vdash \mathtt{x} = \mathtt{x}' \in \mathit{Label}}{\Gamma \vdash (r.\mathtt{x} := a) = (r'.\mathtt{x}' := a') \in \{\mathtt{x} : A\}}$$

$$\frac{\Gamma \vdash r = r' \in R_1 \qquad \Gamma \vdash r = r' \in R_2}{\Gamma \vdash r = r' \in \{R_1; R_2\}}$$

In particular, we can prove that

$$\{\mathtt{x} = 0; \mathtt{y} = 0; \mathtt{color} = \mathit{red}\} = \\ \{\mathtt{x} = 0; \mathtt{y} = 0; \mathtt{color} = \mathit{green}\} \in \{\mathtt{x} : \mathbb{Z}; \mathtt{y} : \mathbb{Z}\}$$

We can also derive the following subtyping properties:

$$\{R_1; R_2\} \subseteq R_1 \\ \{R_1; R_2\} \subseteq R_2 \\ \{R_1; R_2[\mathit{self}]\} \subseteq R_1 \\ \{x : \mathtt{x} : A; R[x]\} \subseteq \{\mathtt{x} : A\}$$

$$\frac{\vdash R_1 \subseteq R_1' \qquad \mathit{self} : R_1 \vdash R_2[\mathit{self}] \subseteq R_2'[\mathit{self}]}{\vdash \{R_1; R_2[\mathit{self}]\} \subseteq \{R_1'; R_2'[\mathit{self}]\}}$$

$$\frac{\vdash A \subseteq A' \qquad x : A \vdash R[x] \subseteq R'[x]}{\vdash \{x : \mathtt{x} : A; R[x]\} \subseteq \{x : \mathtt{x} : A'; R'[x]\}}$$

Further, we can establish two facts that states the equality of left and right associating records.

$\{x : \mathtt{x} : A; R[x]\} =_e \{\mathtt{x} : A; R[\mathit{self}.\mathtt{x}]\}$

$\{R_1; \{x : \mathtt{x} : A[\mathit{self}]; R_2[\mathit{self}, x]\}\} =_e$
$\{\{R_1; \mathtt{x} : A[\mathit{self}]\}; R_2[\mathit{self}, \mathit{self}.\mathtt{x}]\}$

For example, using these two equalities we can prove that

$\{\mathtt{x} : A; \mathtt{y} : B[\mathit{self}.\mathtt{x}]; \mathtt{z} : C[\mathit{self}.\mathtt{x}; \mathit{self}.\mathtt{y}]\} =_e$
$\{x : \mathtt{x} : A; y : \mathtt{y} : B[x]; \mathtt{z} : C[x; y]\}$

4.3 Examples

4.3.1 Semigroup Example

Now we can define the $SemigroupSig$ type in two ways:

$$\{\mathtt{car} : \mathbb{U}; \mathtt{product} : \mathtt{car} \times \mathtt{car} \to \mathtt{car}\} \quad \text{or} \\ \{\mathit{car} : \mathtt{car} : \mathbb{U}; \mathtt{product} : \mathit{car} \times \mathit{car} \to \mathit{car}\}$$

Note that in the first definition `car` in the declaration of `product` stands for $\mathit{self}.\mathtt{car}$, and in the second definition *car* is just a bound variable.

We can define $Semigroup$ by extending $SemigroupSig$:

$$\{SemigroupSig; \mathtt{axm} : \forall x, y, z : \mathtt{car} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)\}$$

or as a right associating record:

$$\{\mathit{car} : \mathtt{car} : \mathbb{U}; \\ \mathit{product} : \mathtt{product} : \mathit{car} \times \mathit{car} \to \mathit{car}; \\ \mathtt{axm} : \forall x, y, z : \mathit{car} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)\}$$

In the first case $x \cdot y$ stands for $\mathit{self}.\mathtt{product}(x, y)$ and in the second case for just $\mathit{product}(x, y)$.

4.3.2 Multiply Inheriting Example

A monoid is a semigroup with a unit. So,

$$MonoidSig \triangleq \{SemigroupSig; \mathtt{unit} : \mathtt{car}\}$$

A monoid is an element of $MonoidSig$ which satisfies the axiom of semigroups and an additional property of the unit. That is, $Monoid$ inherits fields from both $MonoidSig$ and $Semigroup$. We can define the $Monoid$ type as follows:

$$Monoid \triangleq \{\{ MonoidSig; Semigroup; \\ \mathtt{unit\_axm} : \forall x : \mathtt{car} \quad x \cdot \mathtt{unit} = x\}$$

Note, that since $MonoidSig$ and $Semigroup$ shared the fields `car` and `product`, these two fields present in the definition of $Monoid$ twice. This does not create problems, since we allow repeating labels (Section 3.1.2).

Now we have the following subtyping relations:

$$\begin{array}{ccc} SemigroupSig & \supset & MonoidSig \\ \cup & & \cup \\ Semigroup & \supset & Monoid \end{array}$$

4.3.3 Abstract Data Type

We can also represent abstract data types as dependent records. Consider for example a data structure *collection of element of type A*. This data structure consists of an abstract type `car` for collections of elements of the type A, a constant of this type `empty` to construct an empty collection, and functions `member` s a to inquire if element a is in collection s, and `insert` s a to add element a into collection s. These functions should satisfy certain properties that guarantee their intended behavior:

1. The empty collection does not have elements.

2. `insert` s a has all element that s has and element a and nothing more.

A formal definition of the data structure of collections could be written as a record:

$$
\begin{array}{l}
Collection(A) \triangleq \\
\{\mathtt{car} : \mathbb{U}; \\
\ \mathtt{empty} : \mathtt{car}; \\
\ \mathtt{member} : \mathtt{car} \to A \to \mathrm{Boolean}; \\
\ \mathtt{insert} : \mathtt{car} \to A \to \mathtt{car}; \\
\ \mathtt{emp\_axm} : \forall a : A \quad a \notin \mathtt{empty} \\
\ \mathtt{ins\_axm} : \forall s : \mathtt{car} \quad \forall a, b : A \quad (\mathtt{member}\ (\mathtt{insert}\ s\ a)\ b) \\
\qquad \iff (\mathtt{member}\ s\ b) \vee (a = b \in A)\}
\end{array}
$$

5 Sets and Dependent Intersections

Set type constructor allows us to hide a part of a witness.

Example 17 *Instead of defining $Semigroup$ type as an extension of $SemigroupSig$ type with an additional field* axm*, we could define the $Semigroup$ type as a subset of $SemigroupSig$:*

$$Semigroup \triangleq \{S : SemigroupSig \,|\, \forall x, y, z : S.\mathtt{car} \ldots\}$$

Now we will show that the set type constructor (which is primitive in NuPRL) may be defined as a dependent intersection as well.

First, we assume that our type theory has the Top type, that is a supertype of any other type. We will need only one property of the Top type: $T \cap Top = T$ for any type T. (In NuPRL Top is defined as $\bigcap_{x:Void} Void$, where $Void$ is the empty type).

Now consider the following type (squash operator):

$$[P] \triangleq \{x : \mathrm{Top} \mid P\}$$

$[P]$ is an empty type when P is false, and is equal to Top when P is true.

Theorem 18

$$\{x : T \mid P[x]\} =_e x : T \cap [P[x]] \tag{13}$$

We can not take (13) as a definition of sets yet, because we defined squash operator as a set. But actually the squash operator is defined in MetaPRL's version of the NuPRL type theory as a primitive constructor and rules for the set type depend on the squash operator. (See [17] for the rules for the squash type and explanations why this is a primitive type). Thus, we can take (13) as a definition.

Moreover, the squash operator could be defined using other primitives. For example, one can define the squash type using union:

$$[P] \triangleq \bigcup_{x:P} \mathrm{Top}.$$

(Union is a type that dual to intersection [18, 12]).

Remark In is interesting to note that in the presence of Markov's principle [14] there is an alternative way to define $[P]$:

$$[P] \triangleq ((P \equiv\!> \mathrm{Void}) \equiv\!> \mathrm{Void})$$

where $A \equiv\!> B \triangleq \bigcap_{x:A} B$. We will not give any details here, since it is beyond the scope of the paper.

We can also define sets without Top and squash type. First, define *independent* sets:

$$\{A|B\} \triangleq \bigcup_{x:B} A.$$

Then define set type:

$$\{x : A|B[x]\} \triangleq x : A \cap \{A|B[x]\}.$$

The Mystery of Notations It is very surprising that braces $\{\ldots\}$ were used for sets and for records independently for a long time. But now it turns out that sets and records are almost the same thing, namely, dependent intersection! Compare the definitions for sets and records:

$$
\begin{array}{rcl}
\{x : T| \quad P[x]\} & \triangleq & x : T \quad \cap [P[x]] \\
\{self : R_1; \quad R_2[self]\} & \triangleq & self : R_1 \quad \cap R_2[self]
\end{array}
$$

The only differences between them are that we use squash in the first definition and write "|" for sets and ";" for records.

So, we will use the following definitions for records:

$\{self : R_1 \mid R_2[self]\} \triangleq \{self : R_1; [R_2[self]]\} = self : R_1 \cap [R_2[self]]$

$\{x : \mathtt{x} : A \mid R[x]\} \triangleq \{x : \mathtt{x} : A; [R[x]]\} = self : \{\mathtt{x} : A\} \cap [R[self.\mathtt{x}]]$

This gives us the right to use the shortening notations as in Section 4.1.2 to omit inner braces and "*self*". For example, we can rewrite the definition of the $Semigroup$ type as

$$
\begin{array}{l}
Semigroup \triangleq \{\mathtt{car} : \mathbb{U}; \\
\qquad \mathtt{product} : \mathtt{car} \times \mathtt{car} \to \mathtt{car} \mid \\
\qquad \forall x, y, z : \mathtt{car} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)\}
\end{array}
$$

Remark Note that we cannot define dependent intersection as a set:

$$x : A \cap B[x] \triangleq \{x : A \mid x \in B[x]\}. \qquad \text{(wrong!)}$$

First of all, this set is not well-formed in the NuPRL type theory (this set would be a well-formed type, only when $x \in B[x]$ is a type for all $x \in A$, but the membership is a well-formed type in the NuPRL type theory, only when it is true). Second, this set type does not have the expected equivalence relation. Two elements are equal in this set type, when they are equal just in A, but to be equal in the intersection they must be equal in both types A and B (see Example 1).

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