On continuity of computable real functions

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Plan:

1. Numbered sets
2. Computable reals
3. Computable mappings
4. Listable sets
5. Main Lemma
7. Continuity of real computable functions
8. Effective continuity

1. Numbered sets

Numbered set *is a pair* (X, α) *where* X *is a set and* α *is a partial mapping from natural numbers* **N** *onto* X*.*

$$\mathbf{N} \supseteq P \overset{\alpha}{\longmapsto} X$$

α is a **numbering** of X,

i is a number (or **index**) of x, $x = \alpha(i)$

Examples:

- $\mathbf{N}^2$ with $i \longrightarrow (\mathit{left}(i), \mathit{right}(i))$
- rationals **Q** with $i \longrightarrow \mathbf{r}(i) = (-1)^{\mathit{sign}(i)} \mathit{num}(i)/\mathit{den}(i)$
- recursive functions with Gödel numbering $i \longrightarrow \varphi_i$
- r.e. sets with numbering $i \longrightarrow W_i$, $(W_i = \mathit{Dom}\varphi_i)$
- total recursive functions with Gödel numbering

(in this case the set of indices is not even r.e.!)

2. Computable reals

A computable real is defined below as the limit of a computable sequence of rationals which is fundamental in some standard way

A real number a is a **computable real** *if there is a recursive function φ_i such that for all $n, m \in \mathbf{N}$*

$$|\mathbf{r}(\varphi_i(n)) - \mathbf{r}(\varphi_i(n+m))| < 2^{-n}$$

and

$$a = \lim \mathbf{r}(\varphi_i(n))$$

Let γ be the numbering $\gamma : i \longrightarrow \lim \mathbf{r}(\varphi_i(n))$. Then $(\mathbf{CR}, \gamma)$ is a numbered set.

$\mathbf{CR}$ is a field with respect to the usual $0, 1, +, \cdot$

A computable sequence $c_0, c_1, c_2, \ldots$ *in* **CR** *is* **computably fundamental** *if*

$$\forall n, m \left| c_n - c_{n+m} \right| < 2^{-n}$$

Theorem: *Every computably fundamental sequence in* **CR** *converges. Moreover, the limit computable real can be found effectively*

Proof. Limit algorithm: the limit computable real is specified by the sequence of rationals $r_0, r_1, r_2, \ldots$ such that r_n is $n+2$nd rational approximation to c_{n+2}.

Let l be a recursive function which given an index i of a computable fundamental sequence of reals returns its limit $l(i)$.

3. Computable mappings

Let (X, α) *and* (Y, β) *be numbered sets. A mapping* Ψ *from* X *to* Y *is a* **computable mapping** *from* (X, α) *to* (Y, β) *if there is a recursive function* f *from* N *to* N *such that*

1. $\forall n \in \mathit{Dom}\alpha\ (!f(n) \Rightarrow f(n) \in \mathit{Dom}\beta)$
2. $\forall m, n \in \mathit{Dom}\alpha\ (\alpha(m) = \alpha(n) \& !f(m) \Rightarrow$
. $\Rightarrow !f(n) \& \beta(f(m)) = \beta(f(n)))$
3. $\Psi(\alpha(m)) \simeq \beta(f(m))$

The latter condition yields that the following diagram is commutative provided all the arrows are defined:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & Y \\ \alpha\uparrow & & \uparrow\beta \\ N & \xrightarrow{f} & N \end{array}$$

Real computable function *is a computable mapping from* $(\mathbf{CR}, \gamma)$ *to itself.*

Some computable functions on $(\mathbf{CR}, \gamma)$: addition, multiplication, power, sin, etc.

4. Listable sets

A subset $\mathcal{L} \subseteq X$ *for a numbered set* (X, α) *is* **listable** *if for some r.e. set* W_i

$$\alpha^{-1}(\mathcal{L}) = W_i \cap \mathit{Dom}\alpha$$

Index set of $\mathcal{L}$ is "potentially" r.e.

Some properties of listable sets:

- $\emptyset$, X are listable
- finite intersections and r.e. unions of listable sets are listable
- domain of every computable mapping

(Cf. X is r.e. $\Leftrightarrow$ $X{=}\mathit{Dom}(g)$)

- inverse images of a listable set under computable mappings

Listable sets of computable reals:

- $\{x \in \mathbf{CR} \mid x > 0\}$,
- open intervals
- $\{x \in \mathbf{CR} \mid x \neq 0\}$, etc.

Non listable: closed intervals, singletons, rationals, algebraic reals, irrational **CR**'s, etc.

5. Main Lemma

Set of rationals $\mathbf{Q}$ is dense in $\mathbf{CR}$: every open interval in $\mathbf{CR}$ contains a rational.

Lemma *Every nonempty listable set of computable reals contains a rational*

Proof. Let $\mathcal{L}$ be a listable set of computable reals, i.e. for some $i \in \mathbf{N}$

$$\gamma^{-1}(\mathcal{L}) = W_i \cap \mathit{Dom}\gamma$$

Let $x \in \mathcal{L}$ and

$$q(0), q(1), q(2), \ldots \longrightarrow x$$

be a computable fundamental sequence of rationals converging to x. Let K be the standard recursively enumerable nonrecursive set of natural numbers

$$K = \{n \mid n\text{-}\mathit{th\ program\ applied\ to}\ n\ \mathit{terminates}\}$$

For each n define a fundamental sequence ζ_n

$$\zeta_n(k) = \begin{cases} q(m), & \textit{if } n\textit{-th program applied to } n \\ & \textit{terminates in exactly } m \leq k \textit{ steps} \\ & \\ q(k), & \textit{otherwise.} \end{cases}$$

if $n \notin K$ then

$$\zeta_n = q(0), q(1), q(2), \ldots \longrightarrow x$$

if $n \in K$ then

$$\zeta_n = q(0), q(1), \ldots, q(m), \ldots, q(m), \ldots \longrightarrow q(m)$$

Therefore ζ_n converges for each n. Let $h(n)$ be the effectively computed index of the limit point of ζ_n. It is clear that

$$n \notin K \;\Rightarrow\; h(n) \in W_i.$$

Note that "$\Leftarrow$" does not hold since this would give a negative test for K. Therefore for some $n \in K$ $h(n) \in W_i$ which gives a rational $q(m)$ in $\mathcal{L}$

Corollary of the proof *Let $\mathcal{L}$ be a listable set of computable reals and $x \in \mathcal{L}$. Then each sequence of rationals converging to x has a member from $\mathcal{L}$*

6. Continuity of real functions

Theorem *Every real computable function is continuous*

Proof. Continuity of $\Psi(x)$ at $a \in Dom\Psi$:

$$\lim_{x \to a} \Psi(x) = \Psi(a)$$

or in "ϵ-δ language":

$$\forall k \exists m \forall x (|x - a| < 2^{-m} \& !\Psi(x) \Rightarrow |\Psi(x) - \Psi(a)| < 2^{-k})$$

Suppose the opposite holds, i.e.

$$\exists k \forall m \exists x (|x - a| < 2^{-m} \& !\Psi(x) \& |\Psi(x) - \Psi(a)| \geq 2^{-k})$$

Pick such k and consider a set

$$Y = \{y \,|\, !\Psi(y) \;\&\; |\Psi(y) - \Psi(a)| > 2^{-k-1})\}.$$

Y is listable as an inverse image of a listable set

$$\{u \mid |u - \Psi(a)| > 2^{-k-1})\}.$$

Notation: $S(a, r)$ = interval with center a and radius r

By the assumptions made, the sets $U_m = Y \cap S(a, 2^{-m-1})$ for all m are nonempty and listable. By the main lemma, in each U_m we can effectively find a rational point q_m. Moreover, since $|q_m - a| < 2^{-m-1}$ the sequence $\{q_0, q_1, q_2, \ldots\}$ is a computable fundamental one converging to a. Consider the set

$$V = \{v \mid !\Psi(v) \& |\Psi(v) - \Psi(a)| < 2^{-k-1})\}$$

V is a listable set (as the inverse image of a listable one) containing a. By the Main Lemma, $q_n \in V$ for some n, i.e. $!\Psi(q_n)$ and $|\Psi(r_n) - \Psi(a)| < 2^{-k-1}$ which contradicts the choice of $r_n \in U_n$.

7. Effective continuity

Theorem (Separation Theorem) *Given indices of disjoint listable sets $\mathcal{L}_1$ and $\mathcal{L}_2$ of computable reals and $a \in \mathcal{L}_1$ one can effectively find n such that*

$$S(a, 2^{-n}) \cap \mathcal{L}_2 = \emptyset.$$

Effective continuity of Ψ means that there is a total computable function g such that for each $x, a \in \mathbf{CR}$ and for all n

$$|x - a| < 2^{-g(n)} \Rightarrow |\Psi(x) - \Psi(a)| < 2^{-n}$$

Theorem (Tsejtin-Moschovakis) *Every real computable function is effectively continuous*