CS421 Assignment 4: 1997

Due Date: October 31, 1997

1. (Heath 4.16) A planet follows an elliptical orbit, which can be represented in a Cartesian (x, y) coordinate system by the equation

$$ay^2 + bxy + cx + dy + e = x^2.$$

As you have seen in Assignment 3, the orbital parameters a, b, c, d, e can be determined by a linear least squares fit. Assume that the following observations of the planet's position are given

x	1.02	0.95	0.87	0.77	0.67	0.56	0.44	0.30	0.16	0.01
y	0.39	0.32	0.27	0.22	0.18	0.15	0.13	0.12	0.13	0.15

   (a) Use the Matlab function **SVD** to compute the singular value decomposition of the 10×5 least squares matrix.

   (b) Use the singular value decomposition to compute the solution to the least squares problem. With the singular values in order of decreasing magnitude, compute the solutions using the first k singular values, $k = 1, \cdots, 5$. For each of the five solutions obtained, print the values for the orbital parameters and also plot the resulting orbits along with the given data points in the (x, y) plane.

   (c) Perturb the input data slightly by adding to each coordinate of each data point a random number uniformly distributed on the interval $[-0.005, 0.005]$. Compute the singular value decomposition of the new least squares matrix, and solve the least squares problem with the perturbed data as in part (b). Compare the new values for the parameters with those previously computed for each value of k. What effect does this difference have on the plot of the orbits? Can you explain this behavior? Which solution would you regard as better: one that fits the data more closely, or one that is less sensitive to small perturbations in the data? Why?

2. (Heath 4.21) If A is an $n \times n$ matrix of rank one, then A must have the form $A = uv^T$ for some nonzero vectors u and v.

   (a) Show that the scalar $u^T v$ is an eigenvalue of A.

   (b) What are the other eigenvalues of A?

   (c) If the power method is applied to A, how many iterations are required for it to converge exactly to the eigenvector corresponding to the dominant eigenvalue?

3. (Heath 4.6) Write a Matlab program implementing the Lanczos method as given in Section 4.3.9, Heath. Test your program using a random symmetric matrix A of order n having eigenvalues $1, 2, \cdots, n$. To generate such a matrix, using **rand** in Matlab, first generate an $n \times n$ matrix B with random entries uniformly distributed on the interval $[0, 1)$ and then compute the QR factorization $B = QR$. Now take $A = QDQ^T$ where $D = \text{diag}(1, \cdots, n)$. The Lanczos algorithm generates only the tridiagonal matrix T_k at iteration k, so you will need to compute its eigenvalues (i.e., the Ritz values γ_i, $i = 1, \cdots, k$) at each iteration by using the Matlab **eig** function. For the purpose of this exercise, run the Lanczos algorithm for a full n iterations. Type help **diag** in Matlab if you are not familiar with this function.

   To see graphically how the Ritz values behave as iteration proceed, construct a plot with the iteration number on the the vertical axis and the Ritz values at each iteration on the horizontal axis. Plot each pair (γ_i, k), $i = 1, \cdots, k$, as a discrete point at each iteration k (see Fig. 4.2, Heath). As iterations proceed and the number of Ritz values grows correspondingly, you should see vertical "tails" of Ritz values converging on the true eigenvalues. Try several values for n, say, $n = 10, 20, \cdots, 50$, making a separate plot for each.