18 Completeness and Compactness of First-Order Tableaux

18.1 Completeness

Proving the completeness of a first-order calculus gives us Gödel's famous completeness result. Gödel proved it for a slightly different proof calculus, and the proof that we will show here goes back to Beth and Hintikka. Let us briefly resume the propositional case.

The key to the completeness proof was the use of Hintikka's lemma, which states that every downward saturated set, finite or not, is satisfiable. We then showed that every open and complete path is in fact a Hintikka sequence. Putting these two things together we reasoned that the root of an open and complete tableau must be satisfiable. Thus a complete tableau for a valid formula cannot be open which means that every tableau for a valid formula will eventually close.

We will prove the first order case along these lines, but have to keep in mind that several things have changed.

- The definition of a valuation now includes quantifiers.
- The definition of Hintikka sets must take γ and δ formulas into account.
- The notion of a complete tableau needs to be adjusted, because there is now the possibility of non-terminating proof attempts.

Fortunately, we can easily make the necessary adjustments and then proceed as before. First, let us define first-order Hintikka sets. A *Hintikka Set for a universe U* is a set S of U-formulas such that for all closed U-formulas A, α, β, γ, and δ the following conditions hold.

$\mathbf{H_0}$ **:** A atomic and $A \in S \mapsto \bar{A} \notin S$

$\mathbf{H_1}$ **:** $\alpha \in S \mapsto \alpha_1 \in S \wedge \alpha_2 \in S$

$\mathbf{H_2}$ **:** $\beta \in S \mapsto \beta_1 \in S \vee \beta_2 \in S$

$\mathbf{H_3}$ **:** $\gamma \in S \mapsto \forall k \in U.\ \gamma(k) \in S$

$\mathbf{H_4}$ **:** $\delta \in S \mapsto \exists k \in U.\ \delta(k) \in S$

The first axiom expresses the openness of S while the other four state that it is downward saturated. Note that because of axiom H_3, Hintikka sets are usually infinite, unless the universe is finite. But the proof of Hintikka's lemma that we discussed a few weeks ago, did not depend on the fact that the set is finite, so it can easily be adapted to the first-order case.

Hintikka Lemma:

$\forall \mathsf{U}{\neq}\emptyset.\ \forall S{:}\mathsf{Set}(\mathsf{Form}_U).\ \mathsf{Hintikka}(S) \mapsto \exists \mathsf{I}{:}\mathsf{Pred}_S{\to}\mathsf{Rel}(\mathsf{U}).\ \mathsf{U,I}{\models}S$

proof: *Because of axiom* H_0 *we know how to define an intepretation that satisfies all the atomic formulas in* S*.*

Define $\mathsf{I}(P(k_1,...,k_n)) = \begin{cases} \mathtt{f} & \text{if } FP(k_1,...,k_n) \in S \\ \mathtt{t} & \text{otherwise} \end{cases}$

Then I clearly maps all the predicate symbols in S to relations over U. What remains to be shown is $\forall \mathsf{Y}{\in}S.\ \mathsf{U,I}{\models}\mathsf{Y}$. As before, we prove this by structural induction on formulas, keeping in mind that the cases for γ and δ are straightforward generalizations of those for α and β.

base case: If Y is an atomic formula then by definition $\mathsf{Y}{\in}S \mapsto \mathsf{I}(\mathsf{Y}){=}\mathsf{t} \mapsto \mathsf{U,I}{\models}\mathsf{Y}$.

step case: Assume the the claim holds for all subformulas of Y.

- If Y is of type α then $\alpha_1,\alpha_2{\in}S$, hence $\mathsf{U,I}{\models}\alpha_1$ and $\mathsf{U,I}{\models}\alpha_2$. By definition of first-order valuations $\mathsf{U,I}{\models}\mathsf{Y}$.
- If Y is of type β then $\beta_1{\in}S$ or $\beta_2{\in}S$, hence $\mathsf{U,I}{\models}\beta_1$ or $\mathsf{U,I}{\models}\beta_2$ and thus $\mathsf{U,I}{\models}\mathsf{Y}$.
- If Y is of type γ then $\gamma(k){\in}S$ for all $k{\in}U$, hence by induction $\mathsf{U,I}{\models}\gamma(k)$ for all k and by definition of first-order valuations $\mathsf{U,I}{\models}\mathsf{Y}$.
- If Y is of type δ then $\delta(k){\in}S$ for some $k{\in}U$, hence by induction $\mathsf{U,I}{\models}\delta(k)$ for some k and by definition of first-order valuations $\mathsf{U,I}{\models}\mathsf{Y}$.

Now what about the completeness of a tableau? In the propositional case, this meant that the tableau cannot be extended any further, because all formulas have been decomposed. Since the propositional tableau method terminates after finitely many steps, this was an easy thing to define. In the first-order case, however, we have to be a bit more careful.

We know that because of γ-formulas, proofs may have infinite branches. But that is not the main problem, since Hintikka's lemma also works for infinite sets. However, not every infinite branch in a tableau is automatically a Hintikka set.

Consider for example, the formula `∃x,y.P(x,y)`, which is certainly not valid. Thus F`∃x,y.P(x,y)` is satisfiable and because of the correctness of the tableau method we know that every proof attempt will fail. But does *every* failing proof attempt actually give us the Hintikka set that we need to reason that F`∃x,y.P(x,y)` *must* be satisfiable?

Certainy not. Just imagine we start decomposing the main formula, which is a γ formula, over and over again. Then we can go on and on forever without ever touching the inner γ formula and we get an infinite branch that does not satisfy the third Hintikka axiom for this inner γ formula.

So our completeness proof cannot rely on an arbitrary attempt to find a tableau proof. After all, completeness only says that it must be possible to prove every valid formula correct with the tableau method but it doesn't require that *any* attempt will succeed. And the fact that we weren't able to find a proof with a not so bright approach doesn't mean that there is none at all.

However, we can design a more systematic approach that is guaranteed to find a tableau proof if there is one. And then we will show that this systematic method will always find a proof if the formula is valid.

For the systematic method we only have to worry about a treatment of γ formulas that guarantees axiom H_3, since the α, β, and δ rules make sure that the other Hintikka axioms satisfied.

Q: *How can we make sure that all γ formulas are eventually covered completely?*

Well, we have to proceed similarly to an enumeration of lists of integers. We modify the extension procedure for tableaux in a way that each γ formula, and thus every other formula as well, will be revisited on a regular basis.

A systematic procedure for proving a first-order formula X:

Start with the signed formula FX and recursively extend the tableau as follows:

- If the tableau is already closed then stop. The formula is valid.
- Otherwise select a node Y in the tableau that is of *minimal level* wrt. the still unused nodes and extend *every* open branch θ through Y as follows:
  - If Y is α extend θ to $\theta \cup \{\alpha_1, \alpha_2\}$.
  - If Y is β, extend θ to two branches $\theta \cup \{\beta_1\}$ and $\theta \cup \{\beta_2\}$.
  - If Y is γ, extend θ to $\theta \cup \{\gamma(a), \gamma\}$, where a is the first parameter that does not yet occur on θ.
  - If Y is δ, extend θ to $\theta \cup \{\delta(a)\}$, where a is the first parameter that does not yet occur in the tableau tree.

Thus the procedure always copies a γ formula to the end of a branch when it is being considered. This way we make sure that it is considered over and over again, but that all the other formulas on the branch are decomposed before that. Thus in the end all the formulas are being used, because we have only denumerably many parameters. This method is certainly not the most efficient one, but it works.

Using the systematic procedure we can give a new definition of complete tableau. A systematic tableau is called *finished*, if it is either infinite or finite and cannot be extended any further. With this definition we immediately get the following result.

Lemma:

In every finished systematic tableau, every open branch is a Hintikka sequence.

A detailed proof for this lemma would have to show by structural induction that the systematic method does in fact cover all formulas as required in the Hintikka axioms. Together with Hintikka's lemma we get.

Corollary:

In every finished systematic tableau, every open branch is uniformly satisfiable.

The completeness theorem is now an immediate consequence as before.

Theorem (Completeness theorem for first-order logic):

If a first-order formula X is valid, then X is provable. Furthermore the systematic tableau method will construct a closed tableau for FX after finitely many steps.

The first statement follows from the above corollary by contraposition and the fact that the systematic tableau method always "constructs" a finished tableau. As for the second, a closed tableau can only have finite branches, which – according to König's lemma – means that it must be finite.

Note that correctness and completeness is preserved again if we require an *atomically closed tableau*, i.e. a tableau where branches only close if there is an atomc formula and its conjugate. Correctness follows from the fact that an atomically closed tableau is certainly a closed tableau, while the systematic tableau method makes sure that we construct a Hintikka sequence if the tableau does not close (which is the case if it does not close atomically). Hintikka's lemma thus implies

Corollary:

If a first-order formula X is valid, then X the there is an atomically closed tableau for FX.

The corollary also has a second important consequence that will be relevant for the compactness of first-order logic.

Theorem (Löwenheim theorem for first-order logic):

If a first-order formula X is satisfiable, then it is satisfiable in a denumerable domain.

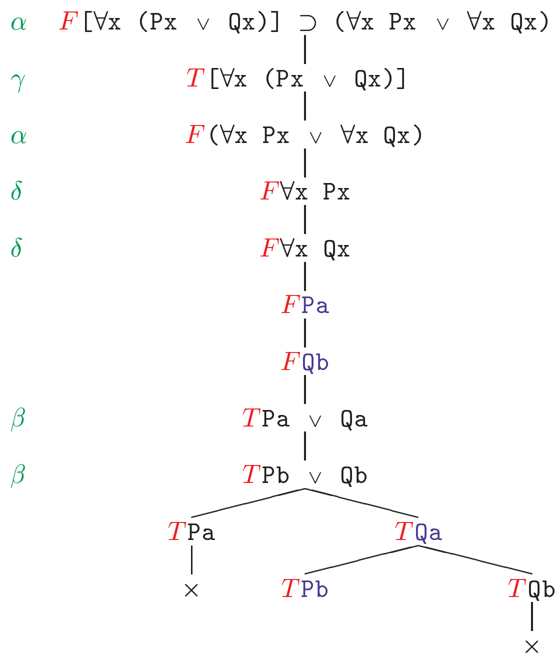
The proof for this theorem is based on the observation that the systematic tableau method uses only denumerably many parameters to build a Hintikka sequence if the tableau doesn't close. Since a tableau with a satisfiable formula at its root cannot close, it must contain an open branch θ with at most denumerably many parameters. As this branch is uniformly satisfiable it satisfies X in a denumerable domain (the subset of the domain U that represents the set of parameters on θ).

18.2 Decidability

While in propositional logic the tableau method could be used as decision procedure, this will certainly not work in first-order logic anymore. If a formula is not valid, the systematic method may lead to an infinite tableau. This is, however, not a deficiency of the tableau method. In fact, there is no correct and complete proof method for first-order logic that always terminates, as first-order logic is known to undecidable.

Nevertheless in some cases, the tableau method can decide that a formula is invalid although the proof is not finished yet. Whenever we have constructed a branch θ that represents a Hintkka set (over the finite domain of the parameters that occur on θ), then we know that the origin FX of the tableau is satisfiable and hence X must be invalid. In these rare cases, the Hintikka branch gives us a *counterexample* for the validity of the formula.

Example: Consider the invalid formula `[∀x (Px ∨ Qx)] ⊃ (∀x Px ∨ ∀x Qx)`.



The tableau to the left cannot be extended anymore in any meaningful way and has one open branch θ, which is a Hintikka set for the 2 element domain U = {a,b}. In this branch the marked α, β, γ, and δ points are fulfilled for the domain U, since all formulas are true under the atomic valuation that assigns `t` to `Qa` and `Pb` and `f` to `Qb` and `Pa`.
In particular, the original formula `[∀x (Px ∨ Qx)] ⊃ (∀x Px ∨ ∀x Qx)` evaluates to `f` under this interpretation.

It is possible to build this "Hintikka Test" into the tableau method and use it to prove that certain formulas cannot be valid. However, there are many formulas that are neither valid nor falsifiable in any finite domain. Any tableau proof attempt for these will run infinitely and at no stage of the proof will we know whether the formula is valid or not.

There are also formulas that are falsifiable, but not in any finite domain. However, the tableau method is "finite" and therefore not suited to produce counterexamples for them.

18.3 Compactness

The last important property of first-order logic that we have to investigate is compactness: *Given a set F of first-order formulas, what does the satisfiability of finite subsets tell us about the satisfiability of the whole set.* In propositional logic we have shown that a set S is uniformly satisfiable if all of its finite subsets are. We gave three proofs for that: one using tableau proofs and König's lemma, one giving a direct construction of a Hintikka set, and one using Lindenbaum's construction, extending S to a maximally consistent set, which turned out to be a proof set.

In first-order logic the question of compactness leads to a spin-off question. Can we extend Löwenheim's theorem to sets of formulas and prove *If a set of formulas is satisfiable, is it satisfiable in a denumerable domain?* – this is the so-called *Skolem-Löwenheim theorem*. So does compactness give us uniform satisfiability over a denumerable domain if we know that all finite subsets are satisfiable? We will give positive answers to all these questions and use the method of tableau for this purpose.

Recall that in the propositional case, we proved compactness by systematically constructing an tableau for the set S, using the fact that every finite subset of S is satisfiable to ensure that the tableau is infinite. We then used König's lemma to show that this tableau has an infinite branch, which in turn must be a Hintikka set. The construction of the tableau made sure that S is a subset of that set and hence satisfiable.

We will proceed in a similar way for first-order logic. First, we define a *first-order tableau for a set S of pure formulas* (i.e formulas without parameters). Such a tableau starts with an arbitrary element of S at its origin and is then constructed by applying one of the 4 rules α, β, γ, or δ, or by adding another element of S to the end of an open branch. The elements of S so added are called the *premises* of the tableau. We call a tableau *complete*, if every open branch is a Hintikka set for the universe of parameters and contains all the elements of S. Obviously every closed tableau is complete as well.
We first show that a complete tableau can be constructed for every set of first-order formulas.

Lemma:

For every denumerable set S there is a complete tableau for S.

Proof: We construct the desired tableau by combining our systematic proof procedure with the construction of a tableau for S that we used in the propositional case. Arrange S as a denumerable sequence $X_1, X_2, X_3, \ldots, x_n, \ldots$.

We begin by placing X_1 at the origin of the tableau. This concludes stage 1. In stage $n+1$ we extend the tableau constructed at stage n as follows.

- If the tableau is already closed then stop. The formula is valid.
- Otherwise select a node Y in the tableau that is of *minimal level* wrt. the still unused nodes and extend *every* open branch θ through Y as in the systematic procedure and add X_{n+1} to the end of every open branch.

By construction every open branch in the resulting tableau is a Hintikka set for the universe of parameters (we used the systematic method) and contains the set S.

Lemma:

If a pure set S has a closed tableau, then a finite subset of S is unsatisfiable.

Proof: Assume S has a closed tableau $\mathcal{T}$ and consider the set S_p of premises of $\mathcal{T}$. By König's lemma, $\mathcal{T}$ must be finite and so is S_p. S_p must be unsatisfiable, since otherwise every branch containing S_p would be open (recall that by construction the elements of the branch are derived from the formulas in S_p using tableau rules only.)

Theorem:

If all finite subsets of a denumerable set S of pure formulas are satisfiable, then S is uniformly satisfiable in a denumerable domain.

Proof: Let $\mathcal{T}$ be a complete tableau for S. Since all finite subsets of S are satisfiable, $\mathcal{T}$ cannot be closed due to the above lemma, so it has an open branch θ. Since $\mathcal{T}$ is complete, θ is a Hintikka for the denumerable universe of parameters that contains S. Thus S is uniformly satisfiable in a denumerable universe.

Corollary: (**Compactness of First-Order Logic**)

If all finite subsets of a pure set S are satisfiable, then S is uniformly satisfiable

Corollary: (**Skolem-Löwenheim theorem for First-Order Logic**)

If a pure set S of is satisfiable then it is satisfiable in a denumerable domain.

Corollary:

If no tableau for a pure set S can close, then S is satisfiable in a denumerable domain.

The last corollary leads to a lot of interesting results about theoretical properties of first-order logic that we won't discuss in this course. Those of you who are interested may study Smullyan's chapters VI and VII.