17 Correctness of First-Order Tableaux

Before the break, we introduced the tableau calculus for first-order logic. It uses signed first-order formulas, and builds a tableau as a dyadic ordered tree in the same way we used for the construction of propositional tableaux. We start with a tableau for FX, where X is the formula to be proven, and apply tableau rules to *extend* the tableau until each branch is *closed*.

While for propositional tableaux we only have α- or β-rules, the presence of quantifiers makes it necessary to introduce two new kinds of rules, so we have 4 rules altogether

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 \mid \beta_2} \qquad \frac{\gamma}{\gamma(\mathrm{a})} \qquad \frac{\delta}{\delta(\mathrm{a})}\ \text{(proviso)}$$

The formulas $\gamma(\mathrm{a})$ are of the form TA^x_a or FA^x_a where γ is $T\forall xA$ or $F\exists xA$. $\delta(\mathrm{a})$ is FA^x_a or TA^x_a where δ is $F\forall xA$ or $T\exists xA$. The proviso states that a must be new, i.e. hasn't been used on the branch so far.

As before, a path can be closed if it contains a formula and its conjugate and a proof for X is a closed tableau for FX.

The calculus as such is almost as simple as the propositional version, but building proofs with it is not always that easy.

Now that we have introduced a proof calculus for first-order logic we have to address the usual questions again, that always come up when dealing with formal proof systems.

1. *Is the tableau method correct? Can we be sure that a proven formula is in fact valid.*
2. *Is it complete? Can we prove every valid formula with the tableau method?*
3. *Is it decidable? Does it always tell us whether a formula is valid or not?*
4. *What about compactness? What does the satisfiability of finite sets of formulas tell us?*
5. *Are there proof strategies for building first-order tableaux that are more successful or more efficient than others?*

17.1 Correctness

Let us begin with correctness. We need to show that every formula that has a first-order tableau proof is actually valid. In the essence we proceed as in the propositional case. We use contraposition and prove that a tableau is satisfiable and cannot be closed whenever the formula at its origin is satisfiable. This implies that the origin of a closed tableau must be unsatisfiable, and since the origin of a tableau for X is FX we know that X must be valid. So essentially we need to show that

$\forall$X:FORM. $\forall\mathcal{T}$:Tableaux$_X$. $\forall$U$\neq\emptyset$. $\forall$I:Pred$_X\to$Rel(U). U,I$\models$origin($\mathcal{T}$) $\mapsto$ $\exists\theta$:path($\mathcal{T}$). U,I$\models\theta$

where U,I$\models\theta$ $\equiv$ $\forall$Y:S-FORM. Y on θ $\mapsto$ (U,I)$\models$Y. This is similar to what we had in the propositional case. However, `I` is now a first-order valuation over `U` instead of a boolean valuation and the definition of $\models$, in contrast to S-value, is based on the interpretation of quantifiers as well.

The proof of this fact is very similar to the one we had before. It just needs to consider γ and δ formulas in addition to α and β formulas. We use structural induction on tableau trees.

base case: If $\mathcal{T}$ has just a single point, then chose θ = [origin($\mathcal{T}$)].

step case: Assume the statement holds for some $\mathcal{T}$ and let $\mathcal{T}_1$ be a direct extension of $\mathcal{T}$ and U,I such that U,I$\models$origin($\mathcal{T}_1$)

Since origin($\mathcal{T}_1$)=origin($\mathcal{T}$), there is some θ:path($\mathcal{T}$) such that U,I$\models\theta$.

Consider 5 cases - the first 3 are identical to what we had before.

1. If $\mathcal{T}_1$ does *not* extend $\mathcal{T}$ at θ, then $\theta\in$path($\mathcal{T}_1$) and U,I$\models\theta$.
2. If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by some α_i, then we know α on θ and U,I$\models\alpha$, thus U,I$\models\alpha_i$. Choose θ_1=$\theta\circ\alpha_i$ then $\theta_1\in$path($\mathcal{T}_1$) and U,I$\models\theta_1$.
3. If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by β_1 and β_2 then we know β on θ and U,I$\models\beta$, thus U,I$\models\beta_i$ for some i. Choose θ_1=$\theta\circ\beta_i$ then $\theta_1\in$path($\mathcal{T}_1$) and U,I$\models\theta_1$.
4. If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by some $\gamma(a)$, then we know γ on θ and U,I$\models\gamma$, thus U,I$\models\gamma$(a). Choose θ_1=$\theta\circ\gamma(a)$ then $\theta_1\in$path($\mathcal{T}_1$) and U,I$\models\theta_1$.
5. If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by some $\delta(a)$ then we know δ on θ and U,I$\models\delta$. Because of the proviso for the δ-rule we know that the parameter a does not occur in any of the formulas of θ and thus U,I$\models\delta(a)$. Choose θ_1=$\theta\circ\delta(a)$ then $\theta_1\in$path($\mathcal{T}_1$) and U,I$\models\theta_1$.

The key argument of the first 4 cases is what we would consider obvious, so I leave that as an exercise to you. But what about the last? How can we show that

If U,I$\models$ S$\cup\{\delta\}$ *and* a *does not occur in any element of* S *then* U,I$\models$ S$\cup\{\delta$(a)$\}$

Mainly we have to unravel the definitions. We express truth for formulas with parameters by truth on U-formulas, use the definition of first-order valuations on U-formulas and then translate the result back into formulas with parameters. For the latter, we need the fact that a does not occur yet.

Proof: Assume U,I$\models$X for every X$\in$S$\cup\{\delta\}$. By definition of $\models$ for formulas with parameters this means that there is a mapping ϕ:Parm$_{S\cup\{\delta\}}\to$U such that U,I$\models$X$^\phi$ for every X$\in$S$\cup\{\delta\}$.

In particular U,I$\models\delta^\phi$ and δ^ϕ is a pure formula of existential type. Thus by definition U,I$\models\delta^\phi$(k) for at least one k$\in$U. Now define

$$\phi^*(p)(\begin{cases} \mathsf{k} & \text{if } p\text{=}\mathsf{a} \\ \phi(p) & \text{otherwise} \end{cases}$$

Then ϕ^*:Parm$_{S\cup\{\delta,\delta(a)\}}\to$U is well defined since a does not occur in any element of S and U,I$\models\delta$(a)$^{\phi^*}$. Thus by definition of $\models$ U,I$\models$X for every X$\in$S$\cup\{\delta,\delta$(a)$\}$.