8 Compactness (Lindenbaum's Theorem)

8.1 Lindenbaum's Theorem

The proof of the compactness theorem that we are going to study today is quite different from proofs that are based on Hintikka's lemma. In a sense it is more abstract but at the same time it is more "constructive" as well. It's basic idea is to extend a consistent set S into one that is *maximally consistent*, that is into a set S^* that cannot be extended further without becoming inconsistent, and then to *extract* an interpretation that satisfies S from S^*. This construction is due to Lindenbaum and will turn out to be very useful for studying first order logic.

We define a set S to be *maximally consistent* if it is consistent and no proper superset of S is consistent.

An example of a maximally consistent set is a truth set, which can be expressed as the set of formulas that are true under some interpretation $\mathsf{v}_0{:}\mathsf{Var}{\to}\mathbb{B}$ (i.e. $S = \{X \mid \mathsf{Val}(X,\mathsf{v}_0) = \mathsf{t}\}$). A truth set is consistent since it is satisfiable. Furthermore, no formula $Y \notin S$ can be added to S without making it inconsistent, since $\{Y,\bar{Y}\}$ would be a finite subset of the resulting set. We will now show that truth sets are the only sets that are maximally consistent.

Lemma 8.1 *Every maximally consistent set is a truth set.*

Proof: Let S be maximally consistent. We prove that S satisfies the axioms of truth sets

S_0: $X \in S \Leftrightarrow \bar{X} \notin S$

Let X be an arbitrary formula. Since S is consistent, either X or $\bar{X}$ cannot be in S – otherwise $\{X;\bar{X}\}$ would be an unsatisfiable finite subset of S.

Now assume that neither X nor $\bar{X}$ are in S. Because maximality, both X and $\bar{X}$ must be inconsistent with S, which means that there are finite subsets S_1,S_2 of S such that $S_1 \cup \{X\}$ and $S_2 \cup \{\bar{X}\}$ are unsatisfiable. Thus $S_3 = S_1 \cup S_2$ is a finite subset of S and both $S_3 \cup \{X\}$ and $S_3 \cup \{\bar{X}\}$ are unsatisfiable as well. This in turn is only possible if S_3 is already unsatisfiable, which is a contradiction to the consistency of S.

Thus for each formula X exactly one of X and $\bar{X}$ is in S.

S_α: $\alpha \in S \Leftrightarrow \alpha_1 \in S \wedge \alpha_2 \in S$

If $\alpha \in S$ then α_1 and α_2 are consistent with S and thus in S because of maximality.

The argument for consistency is as follows: Let $\alpha \in S$ and S_0 be a finite subset of $S \cup \{\alpha_1,\alpha_2\}$. Then S_0-$\{\alpha_1,\alpha_2\} \cup \{\alpha\}$ is a finite subset of S, hence uniformly satisfied by some $\mathsf{v}_0{:}\mathsf{Var}{\to}\mathbb{B}$. By the laws of Boolean valuations v_0 satisfies α_1 and α_2 as well and therefore S_0 is satisfiable.

Conversely, if α_1 and α_2 are in S then α is consistent with S, hence $\alpha \in S$

S_β: $\beta \in S \Leftrightarrow \beta_1 \in S \vee \beta_2 \in S$

If $\beta \in S$ then either β_1 or β_2 is consistent with S and thus in S because of maximality.

Conversely, if β_1 or β_2 are in S then β is consistent with S, hence $\beta \in S$

Since there is a one-to-one correspondence between truth sets and interpretations, it is easy to extract an satisfying interpretation from a maximally consistent set S. Simply define $\mathsf{v}_0(p) = \begin{cases} \mathtt{t} & \text{if } p \in S \\ \mathtt{f} & \text{if } \bar{p} \in S \end{cases}$ and by construction v_0 satisfies all the formulas in S.

Corollary 8.2 *Every maximally consistent set S is satisfiable.*

What is left to show is that we can extend every consistent set S into a maximally consistent one. Since maximally consistent sets are uniformly satisfiably, S must be satisfiable as well.

The key idea of Lindenbaum's proof of this fact is to add formulas to the set S whenever they are consistent with the resulting set. Obviously this is an infinite process. But since there are only *denumerably many formulas*, we can give a recursive description of a method for construction a maximally consistent set. This construction works because consistent sets have the following property:

A set S is consistent iff all finite subsets of S are consistent

This is easy to see. If S is consistent, then all finite subsets of its finite subsets are satisfiable, so its finite subsets are consistent. On the other hand if all finite subsets are consistent, then the are satisfiable because they are finite subsets of themselves and thus S is consistent.

In fact, this property of consistency is the only thing we need to build a maximally consistent set. Therefore we will prove a slightly more general theorem and derive Lindenbaum's theorem as a corollary.

We say that a property P has *finite character* if for any set $P(S)$ holds if and only if $P(S')$ holds every finite subset S' of S. We will show that whenever a property P has finite character then every set S having property P can be extended into a maximal set that has property P.

Lemma 8.3 (Tukeys Lemma)
Let $U = \{Y_1, Y_2, \ldots\}$ be a denumerable universe, P be a property of finite character on subsets of U, and $S \subseteq U$ with $P(S)$. Then there is a maximal set $S^* \subseteq U$ with $S \subseteq S^*$ and $P(S^*)$.

Proof: We inductively construct an infinite sequence of sets S_i as follows:

$$S_0 := S \quad S_{n+1} := \begin{cases} S_n \cup \{Y_{n+1}\} & \text{if this set has property } P \\ S_n & \text{otherwise} \end{cases}$$

We define $S^* := \bigcup S_i$. Then obviously $S \subseteq S^*$

To prove $P(S^*)$ we use the finite character of P and show that $P(K)$ holds for every finite subset $K \subseteq S^*$.
Let K be a finite subset of S^*. Then $K \subseteq S_i$ for some i. By construction we know that $P(S_i)$ holds. Therefore $P(K)$ holds because of the finite character of P.

To prove maximality, we assume that P holds for some extension $S^* \cup \{Y_i\}$. Consider the set $S_{i-1} \cup \{Y_i\}$. Since each finite subset K of $S_{i-1} \cup \{Y_i\}$ is a finite subset of $S^* \cup \{Y_i\}$ and thus has property P, $S_{i-1} \cup \{Y_i\}$ must have property P as well (using the finite character of P in both directions). By construction this means $Y_i \in S_i$ and thus $Y_i \in S^*$.

Note that both the construction of the set S^* and the proof of its maximality require that we can select elements from a denumerable universe and argue that every element of the universe will be considered at some point of the construction. For nondenumerable universes we need the axiom of choice to prove a general version of Tukeys Lemma. In fact, *Tukeys Lemma is known to be equivalent to the axiom of choice*. However, as as long as the universe of elements from which S can formed is denumerable, we don't need the axiom of choice but are able to give a constructive proof that reflects Lindenbaum's original argument.

Lindenbaum's Theorem is an immediate consequence of the above lemma because consistency is a property of finite character and because the universe of all formulas is denumerable.

Theorem 8.4 (Lindenbaum's Theorem)
Every consistent set of formulas can be extended into a maximally consistent one.

The compactness theorem now follows from Lindenbaum's Theorem and the fact that maximally consistent sets are satisfiable.

Corollary 8.5 *Every infinite consistent set of formulas is satisfiable*

8.2 An Analytic variant of Lindenbaum's theorem

Lindenbaum's proof for the compactness theorem is *synthetic* in the sense that it considers the complete universe of formulas to extend a consistent set S into a maximally consistent one before extracting an interpretation that satisfies S. In contrast to that the proofs that we gave before were *analytic*, because they focus on formulas that *descend* from formulas in S, i.e. subformulas of formulas in S and their conjugates. Most proof methods have to be analytic to guarantee success in a reasonable amount of time. As soon as we allow arbitrary formulas to be used in a proof, the proof method becomes difficult to automate or horribly inefficient.[2]

On the other hand, Lindenbaum's construction is much simpler than the constructions given in the other proofs, which depend on a careful scheme for generating an infinite Hintikka set. It is therefore desirable to modify Lindenbaum's construction into an analytic one.

We call a formula Y a *direct descendent* of X if X is α or β and Y is α_1, α_2, β_1, or β_2. Y is a *descendant* of X if it is identical to X or a direct descendant of some descendant of X. For a set S of formulas we define S^0 to be the set $\{Y{:}\mathsf{Form} \mid \exists X \in S.\ Y \text{ descendant of } X\}$ of all descendants of elements of S. We will use this set, which is obviously a superset of S, instead of the set of all formulas as universe for Lindenbaum's construction.

Because of Lindenbaum's theorem we know that every consistent set S has a maximally consistent extension S^* wrt. the universe S^0. Because of the limitation to S^0 this set is not necessarily a truth set, since it may very well be the case that we can extend it further by formulas that are not in S^0. However, we can still show that it is a Hintikka set.

Lemma 8.6 *Let S be a set of formulas. Then every maximally consistent subset of S^0 is a Hintikka set.*

[2] In some future lecture we will revisit this issue when investigating "cut-free" proofs

The proof is similar to the one of Lemma 8.1. The arguments for downward saturation still hold because the universe is a superset of S.

Proof: Let M be a maximally consistent subset of S^0.

$\mathbf{H_0}$: $p \in M \mapsto \bar{p} \notin M$
This follows directly from the consistency of M.

$\mathbf{H_1}$: $\alpha \in M \mapsto \alpha_1 \in M \wedge \alpha_2 \in M$
If $\alpha \in M$ then α_1 and α_2 are consistent with M and thus in M because of maximality.

$\mathbf{S_\beta}$: $\beta \in \mathrm{S} \Leftrightarrow \beta_1 \in \mathrm{S} \vee \beta_2 \in \mathrm{S}$
If $\beta \in S$ then either β_1 or β_2 is consistent with S and thus in S because of maximality.

Together with Tukey's lemma for the universe S^0 this lemma proves the compactness theorem but extends S only by some of its descendants.

8.3 The Compactness Theorem for Deducibility

There is one, quite useful variant of the compactness theorem addresses the notion of deducibility. We say that a formula *X is deducible from a set S* if for some finitely many $X_i \in S$ the formula $X_1 \wedge .. \wedge X_n \supset X$ is a tautology. Now the compactness theorem relates formal deducibility to the notion of a *logical consequence*.

Theorem 8.7 (Deduction Theorem)
If X is true under all interpretations that satisfy S then X is deducible from S.

This means that a logical consequence of an infinite set of formulas can be proven on the basis of a finite subset of that set.

Proof: Let X is true under all interpretations that satisfy S. Then $S \cup \{\sim X\}$ is unsatisfiable. Because of compactness, there must be a finite subset $S_0 = \{X_1, \ldots, X_n, \sim X\}$ of $S \cup \{\sim X\}$ that is unsatisfiable. This means that $X_1 \wedge .. \wedge X_n \supset X$ must be a tautology.