7 Compactness

7.1 Compactness – why?

So far, we have applied the tableau method to propositional formulas and proved that this method is sufficient and adequate to show that a formula is *valid*, i.e. a tautology. The tableau method is a finite proof method and can be implemented on a computer as a decision procedure for propositional logic.

The obvious question that comes up at this point is: can we go *beyond simple propositional formulas* or are there limitations of such finite proof methods? Can we use them to investigate sets of formulas, even if they are infinite? Can we use finite methods to something infinite? This question is not only mathematically interesting but will also tell us to what extent we can use the method for *first order logic*, when quantifiers will turn a single formula into a description of infinite sets of formulas.

You probably all know that universal quantifiers are nothing but an unlimited conjunction: Instead of $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \ldots$ we can also write $\forall \mathrm{i}.X_i$. In the same way, the existential quantifier can be related to unlimited disjunction: instead of $X_1 \vee X_2 \vee X_3 \vee X_4 \ldots$ we can write $\exists \mathrm{i}.X_i$. So one could imagine extending the tableau method as follows:

> To apply the method to $\exists \mathrm{i}.X_i$ we start with $F(X_1 \vee X_2 \vee X_3 \vee X_4 \ldots)$ and decompose this formula step by step: Since we know that all the FX_i are direct descendants of that formula, we first derive FX_1 and try to close that tableau. If that doesn't work, we derive FX_2, add it to every open branch and try to close the resulting tableau, etc.

It is obvious that we have a proof for $\exists \mathrm{i}.X_i$ if we arrive at a closed tableau with this method. After all, we have shown that $F(X_1 \vee X_2 \vee \ldots \vee X_n)$ is unsatisfiable for some n, so $X_1 \vee X_2 \vee \ldots \vee X_n$ is valid and that certainly implies $\exists \mathrm{i}.X_i$. But, what information do we gain if we cannot close the tableau at all? Does the fact that $F(X_1 \vee X_2 \vee \ldots \vee X_n)$ is satisfiable for any n imply that $F(\exists \mathrm{i}.X_i)$ must be satisfiable as well?

Q: *Why isn't that a trivial question?*

The difficulty here is that we must relate a property of an infinite set of formulas to what we know about its finite subsets. In real analysis, for instance, we know that an infinite set does not necessarily have the same properties as all its finite subsets: every element of the sequence [1/2; 1/3; 1/4; 1/5; ...] is greater than zero, but the limit of that sequence is not.

So, in principle, it could be possible that every interpretation that satisfies some subset $F(X_1 \vee X_2 \vee \ldots \vee X_n)$, will not work on some larger subset of $F(X_1 \vee X_2 \vee \ldots)$ anymore and that the $F\exists \mathrm{i}.X_i$ is unsatisfiable after all – we just used the wrong method in our attempt to prove this.

In mathematics, the question that we just raised relates to the subject of *compactness*: given a possibly infinite set – – what can we say about its properties when we know the properties of its finite subsets?

In logic, *compactness* focuses on *satisfiability* of formulas: *if all finite subsets of a set S of formulas are satisfiable, is there is a valuation v_0:Var $\rightarrow \mathbb{B}$ that satisfies all the formulas in S uniformly?*

Obviously the answer to that question does not depend on the tableau method but only on propositional logic as such. However, the tableau method is one way to investigate it and to prove that the answer to that question is indeed a *yes*, as it relates the notion of satisfiability to finitely generated trees whose open branches are Hintikka sets.

To simplify the notation, we define a set S of formula to be *consistent* if every finite subset of S is satisfiable. The property that we want to prove thus can be formulated as

Theorem 7.1 (Compactness Theorem)
Every infinite consistent set of formulas is satisfiable

This theorem gives us the assurance that finite finite methods like the tableau method can be extended to first-order logic and beyond.

There are many different ways to prove compactness. Most of these have turned out to be so fundamental for the development and study of proof methods, that it is worth looking at a variety of proofs for the same result and try to understand the methodology behind them.

We will look at a proof that uses the tableau method, Hintikka's lemma, and a lemma about finitely generated trees. We will then look at a proof that bypasses the tableau method and works directly with the Hintikka property. And finally, we will study a very abstract and powerful method, called *Lindenbaum's theorem* that analyzes consistency as such.

7.2 A proof based on the tableau method

The first proof for the theorem will be based on the tableau method. Since the issue we're interested in is satisfiability of a set of formulas and since it doesn't matter if these formulas are signed or not, let us consider tableaux for unsigned formulas.

Obviously, a finite set of formulas $\{X_1, \ldots X_n\}$ is uniformly satisfiable if its conjunction $X_1 \wedge \ldots \wedge X_n$ is satisfiable. If this is the case then (because of the correctness of the tableau method) every tableau proof for $X_1 \wedge \ldots \wedge X_n$ must have at least one open branch.

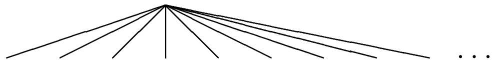
Now consider the construction of a tableau for a consistent set $S = \{X_1, X_2, X_3, \ldots\}$. The construction will analyze the formulas of S in some order. Since every finite subset of S is satisfiable, we're going to have an open branch at every stage of the development, which means the construction will have to go on forever and the resulting tableau will have infinitely many points. Now the question is

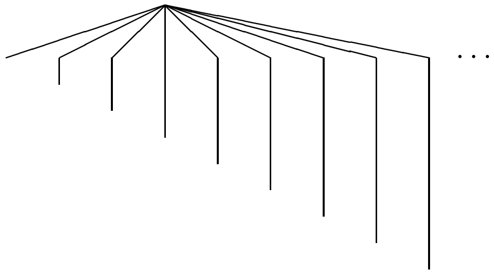
Q: *Does that mean the tableau has an infinite open path?*

Let us simplify the question.

Q: *Does every infinite tree necessarily have an infinite branch?*

Well, the answer obviously depends whether we allow a point to have infinitely many successors or not. If we do allow infinitely many successors the answer is *no*, as the following trivial example (left) shows.





Even if we allow the paths to grow longer and longer, that even if we do not place any length limit on the branches of the tree, then infinitely many points in the tree still do not guarantee the existence of infinite branches, as the example on the right shows. But if we consider *finitely generated trees* – and formula trees and tableaux are of that kind – then there is in fact a uniform length limit. This insight is called

Lemma 7.2 (König's Lemma)
Every infinite, but finitely generated tree has an infinite branch

In the literature you will find two formulations of this lemma, the one given above and another, which is an immediate consequence.

Corollary 7.3 (König's Lemma)
Every finitely generated tree that has only finite branches is finite

We're going to prove the first, since this is the version we will use later.

Proof: Let T be a finitely generated tree and with infinitely many points. We will construct an infinite branch θ in T by considering those points in the tree that have infinitely many descendants. In König's original proof these points are called *good* and we will adapt this notation.

Obviously the origin of T is good and if a point is good and has only finitely many successors, one of them must be good as well.

So $p_0 = \mathsf{origin}(T)$ is good and has a successor p_1 that is good, p_1 has a successor p_2 that is good, etc. Using this argument recursivily we get an infinite chain $[p_0,\ p_1,\ p_2,\ \ldots]$ of points in T that form a branch in T.

Using König's Lemma we will now prove the compactness theorem:

Assume that $S = \{X_1, X_2, X_3, \ldots\}$ is consistent. We construct a tableau tree for S as follows: We begin with a complete tableau for X_1. Since S is consistent, X_1 is satisfiable and the tableau will have at least one open branch. Next, we add the formulas $X_1 \wedge X_2$ and X_2 at the top of the tableau, which will give us an incomplete tableau for $X_1 \wedge X_2$. We continue the development of the tableau for $X_1 \wedge X_2$ at the open branches until it is complete. Again, at least one branch will be open and we continue by adding $X_1 \wedge X_2 \wedge X_3$ and X_3 at the top, etc.

Since at each stage of the development we have a tableau for a finite subset of S, the construction will never terminate and we will get an infinite tree that contains all $X_i \in S$.

Because of König's Lemma, this tree must have an infinite branch and, by construction, all the X_i are on this branch.

Since we used the tableau method, the set S^* of formulas on this branch will be a Hintikka set that is a superset of S. Because of Hintikka's lemma, S^* is satisfiable and so is S as a subset of S^*.

This proof does not only prove the compactness theorem but also explains how we could define a tableau for an infinite set S of formulas: We start with a formula $X \in S$ and apply in each step either one of the tableau rules or add an arbitrary formula from S to the end of the branch. If the tableau is not closed, it will become infinite and thus show the satisfiability of S.

Obviously, this is a purely theoretical approach, since we cannot cannot wait infinitely long to get an answer. However, it illustrates the principles.

7.3 A direct construction of Hintikka sets

The second proof that we're going to look at bypasses the specific proof method of tableau proofs and directly constructs a Hintikka set for any consistent set S. This shows again that compactness does not depend on the method that one may use for proving formulas true. It is a property of propositional logic.

We begin our proof by stating a few observations about consistency, which will then be used in the rest of the argument.

Obviously, *any subset S' of a consistent set S is consistent* since the subsets of S' are subsets of S as well. Secondly, consistency satisfies the following three conditions

C_0: $\forall$P:S-Var$_X$. (P $\in S \wedge \bar{\text{P}} \in S$) $\mapsto$ S inconsistent

C_1: $S \cup \{\alpha\}$ consistent $\mapsto$ $S \cup \{\alpha_1,\alpha_2\}$ consistent[1]

C_2: $S \cup \{\beta\}$ consistent $\mapsto$ $S \cup \{\beta_1\}$ consistent or $S \cup \{\beta_2\}$ consistent

Condition C_0 is obvious since $\{\text{P},\bar{\text{P}}\}$ would be an unsatisfiable subset of S.

To show C_1, we assume that $S \cup \{\alpha_1,\alpha_2\}$ is inconsistent. Then there is a finite subset S_1 of $S \cup \{\alpha_1,\alpha_2\}$ such that $S_1 \cup \{\alpha_1,\alpha_2\}$ is unsatisfiable. Due to the laws of boolean valuations $S_1 \cup \{\alpha\}$ must be unsatisfiable as well, which means that $S \cup \{\alpha\}$ is inconsistent.

The justification for C_2 is similar.

Now let us come back to the proof of the compactness theorem and assume that $S = \{X_1, X_2, X_3, \ldots\}$ is consistent. We will inductively construct a Hintikka set S^* that is a superset of S, which implies the satisfiability of S.

- We start our construction by defining $S_1 := \{X_1\}$
- For the construction of S_{n+1} we assume that $S_n = \{Y_1, \ldots Y_{n+i}\}$ is a superset of $\{X_1, \ldots, X_n\}$ that is *consistent with S* (i.e. $S \cup S_n$ is consistent), and consider the element Y_n.

[1] Smullyan uses the notation $\{S,\alpha_1,\alpha_2\}$ as shorthand for $S \cup \{\alpha_1,\alpha_2\}$

If Y_n is a variable, we define $S_{n+1} := S_n \cup \{X_{n+1}\}$, which is consistent with S.

If Y_n is α, we define $S_{n+1} := S_n \cup \{\alpha_1, \alpha_2, X_{n+1}\}$, which is consistent with S according to C_1.

If Y_n is β, then we define $S_{n+1} := S_n \cup \{\beta_j, X_{n+1}\}$ such that S_{n+1} is consistent with S. Because of C_2 this is the case for at least one of β_1 or β_2

- We define $S^* = \bigcup S_n$.

Then S^* is a superset of S by construction. S^* also satisfies the axioms of a Hintikka set.
– H_0 is satisfied since $\{P, \bar{P}\}$ is not a subset of any $S \cup S_n$, thus not of S^*
– H_1: if $\alpha \in S^*$ then $\alpha = Y_n$ for some n, thus $\alpha_1, \alpha_2 \in S_{n+1} \subseteq S^*$
– H_2: if $\beta \in S^*$ then $\beta = Y_n$ for some n, thus $\beta_j \in S_{n+1} \subseteq S^*$ for some j

Since S^* is a Hintikka set, it is uniformly satisfiable and so is S, because it is a subset of S^*.