6 Completeness and Decidability

6.1 Completeness

Proving completeness of a formal system means showing that our system is strong enough to find a proof for everything that is true. This is usually more complicated that showing correctness, which only requires us to show that we had the right start and that all the rules work correctly. For completeness, we implicitly have to describe a method how to find a proof for a tautological formula.

Our chain of reasoning will now be the following: If X is a tautology, then FX is unsatisfiable. If FX is unsatisfiable then there is a tableaux for it such that every path is closed, which means that X has a tableau proof.

Again, we will proceed by contraposition, showing that FX is satisfiable if every tableau for X has an open path. The central idea is to focus our attention on *complete tableaux*, i.e. tableaux where no path can be extended anymore. If all these have at least one open path, then this path will give us an interpretation for the variables in X that makes FX true, which in turn means that FX is satisfiable. Actually, it suffices to find only one complete and open path in a tableau for X to find an interpretation that makes FX true.

Lemma 6: $\forall \mathsf{X}{:}\mathsf{FORM}.\ \forall \mathcal{T}{:}\mathsf{Tableaux}_X.\ \forall \theta{:}\mathsf{path}(\mathcal{T}).$
$(\mathsf{complete}(\theta) \wedge \mathsf{open}(\theta)) \mapsto \exists \mathsf{v}_0{:}\mathsf{Var}_X{\to}\mathbb{B}.\ \mathsf{true}(\theta,\mathsf{v}_0)$

Corollary 7: $\forall \mathsf{X}{:}\mathsf{FORM}.\ \forall \mathcal{T}{:}\mathsf{Tableaux}_X.\ (\mathsf{complete}(\mathcal{T}) \wedge \mathsf{open}(\mathcal{T})) \mapsto \mathsf{satisfiable}(FX)$

Corollary 8: $\forall \mathsf{X}{:}\mathsf{FORM}.\ X\ \mathsf{tautology} \mapsto \forall \mathcal{T}{:}\mathsf{Tableaux}_X.\ \mathsf{complete}(\mathcal{T}) \mapsto \mathsf{closed}(\mathcal{T})$

Lemma 9: $\forall \mathsf{X}{:}\mathsf{FORM}.\ \exists \mathcal{T}{:}\mathsf{Tableaux}_X.\ \mathsf{complete}(\mathcal{T})$

Theorem 10: $\forall \mathsf{X}{:}\mathsf{FORM}.\ X\ \mathsf{tautology} \mapsto \exists \mathcal{T}{:}\mathsf{Tableaux}_X.\ \mathsf{closed}(\mathcal{T})$

What remains to be shown is lemma 6. Assume, we have a complete and open path θ in a tableau for X. Then θ has the following properties:

H_0: $\forall \mathrm{P}{:}\mathrm{S\text{-}Var}_X.\ \mathrm{P} \in \theta \mapsto \bar{\mathrm{P}} \notin \theta$ — openness of θ

H_1: $\alpha \in \theta \mapsto \alpha_1 \in \theta \wedge \alpha_2 \in \theta$ — completeness of θ

H_2: $\beta \in \theta \mapsto \beta_1 \in \theta \vee \beta_2 \in \theta$ — completeness of θ

Actually, the fact that θ is a path is not relevant for these properties. All we need to consider is the set of formulas in θ. Sets that have these three properties are called *downward saturated* or *Hintikka sets* in honor of the logician, who first studied their properties.

Note that Hintikka sets are very closely related to truth sets (i.e. saturated sets). The difference is that the axioms of truth sets have a bi-implication where Hintikka sets have an implication. They are fully saturated both downward and upward.

S_0: $\forall X{:}\text{SForm}.\ X \in S \Leftrightarrow \bar{X} \notin S$

S_α: $\alpha \in S \Leftrightarrow \alpha_1 \in S \wedge \alpha_2 \in S$

S_β: $\beta \in S \Leftrightarrow \beta_1 \in S \vee \beta_2 \in S$

So what we really want to show is that *every Hintikka set is satisfiable*, or that *every downward saturated set can be extended to a saturated set*, which we formulate as follows.

Hintikka Lemma: $\forall S{:}\mathsf{Set(Form)}.\ \mathsf{Hintikka}(S) \mapsto \exists \mathsf{v}_0{:}\mathsf{Var}\to\mathbb{B}.\ \mathsf{true}(S,\mathsf{v}_0)$

proof: *Satisfiability of S means that all formulas in S must be made true by some interpretation v_0. Because of axiom H_0 we know how to satisfy all the signed variables in S.*

Define $\mathsf{v}_0(\mathsf{p}) = \begin{cases} \mathtt{f} & \text{if } F\mathrm{p} \in S \\ \mathtt{t} & \text{otherwise} \end{cases}$

claim: $\forall \mathsf{Y}{:}\mathsf{FORM}.\ \mathsf{Y} \in S \mapsto \mathsf{S\text{-}value}(\mathsf{Y},\mathsf{v}_0)=\mathsf{t}$

proof: By structural induction on formulas

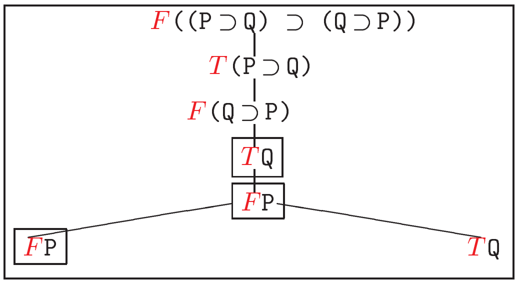
base case: If Y is a signed variable then $\mathsf{Y} \in S \mapsto \mathsf{S\text{-}value}(\mathsf{Y},\mathsf{v}_0)=\mathsf{t}$ by definition

step case: Let Y be a signed formula and assume the the claim holds for all subformulas of Y *(their degree is smaller than the degree of Y)*.
– If Y is of type α then $\alpha_1,\alpha_2 \in S$ (axiom H_1), hence $\mathsf{S\text{-}value}(\alpha_1,\mathsf{v}_0)=\mathsf{t}$ and $\mathsf{S\text{-}value}(\alpha_2,\mathsf{v}_0)=\mathsf{t}$. Therefore $\mathsf{S\text{-}value}(\mathsf{Y},\mathsf{v}_0)=\mathsf{t}$
– If Y is of type β then $\beta_1 \in S$ or $\beta_2 \in S$ (axiom H_2), hence $\mathsf{S\text{-}value}(\beta_1,\mathsf{v}_0)=\mathsf{t}$ or $\mathsf{S\text{-}value}(\beta_2,\mathsf{v}_0)=\mathsf{t}$. Therefore $\mathsf{S\text{-}value}(\mathsf{Y},\mathsf{v}_0)=\mathsf{t}$

Again the main argument is based on the properties of Boolean valuations expressed in uniform notation. The proof would have been much more complex if we had to describe the properties of Boolean valuations in terms of logical connectives. Many proofs in the older literature therefore appear very complicated.

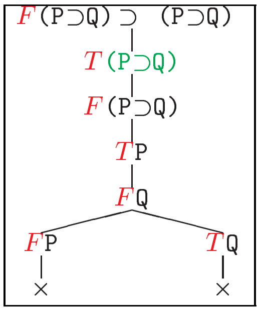
Note that we didn't have to assume that the Hintikka set is finite. Furthermore, it isn't exactly necessary to use signed formulas for the proof as it doesn't depend on that - we only need a notion of α and β formulas as well as the concept of conjugates.

The proof of Hintikka's lemma is more than just evidence for the completeness of the tableau method. It shows us how to find a *counterexample* for a formula X that is not a tautology. We only have to construct a complete tableau for the formula, look at an open path, and use the interpretation v_0 constructed in the proof. This interpretation, as shown, satisfies FX and is therefore a counterexample for X.



We illustrated that in the last lecture. Here is another example of a formula that is sometimes called the *classical misconception*: "If I know that P implies Q and I know that Q holds, then P must be reason for that so P is true." However, that is just a misconception, as the tableau to the right shows.

The conditions for a tableau proof state that a tableau is closed if each branch contains a formula and its conjugate. This means that in some cases we may end a proof fairly early. A tableau proof for `(P⊃Q) ⊃ (P⊃Q)`, for instance, may stop after the first step, because it is already closed.
However, checking for conjugates in a large tableau tree might be time consuming and for this reason there are refinements of the tableau method that check only for conjugates of atomic formulas. This means that we have to go on decomposing a formula until we have signed variables and then check for conjugates.



Obviously requiring that all branches of a tableau are *atomically closed*, i.e. contain a signed variable and its conjugate, doesn't change anything about the correctness of the tableau method,

Q: *Why?*
because an atomically closed tableau is also a closed tableau and the existence of a closed tableau for FX implies that the formula X is a tautology.

Q: *But what about completeness?*
We strengthened the requirements for tableau proofs, so it is possible that we may not always be able to find an atomically closed tableau for a tautological formula.

However, if we look closely at the completeness proof we may realize that it is not affected by that condition at all. The completeness proof is based on *open* paths, and not on the conditions for closed paths. And if a path is open and complete, then it contains decompositions of formulas all the way down to the atoms (that is the definition of completeness) and but no formula and its conjugate (that is axiom H_0). In other words it is a Hintikka set.

So, assume that X is a tautology but that there is no atomically closed tableau for FX. This means that there is a complete tableau for FX that is not atomically closed, which means it must contain at least one complete and (atomically) open path. Because this path is a Hintikka set, it is satisfiable, which means that FX is satisfiable and that X cannot be a tautology.
So our completeness proof actually gave us a stronger result than we originally aimed for.

Corollary 11: $\forall \mathsf{X{:}FORM}.\ X\ \mathsf{tautology} \mapsto \exists \mathcal{T}{:}\mathsf{Tableaux}_X.\ \mathsf{atomically\text{-}closed}(\mathcal{T})$

6.2 Decidability

The tableau method for propositional logic is not only a complete and correct method for constructing proofs but it also helps us *decide* whether a formula is true or false. Smullyan doesn't emphasize this, but since we already know that the method terminates and provides counterexamples if the formula is not a tautology.

Theorem 12: For every formula X we can construct either a tableau proof an interpretation that falsifies X.

proof: Start with the root tableau FX and extend it by applying α and β rules until the tableau is complete. If the tableau is closed, we have found a tableau proof for X. Otherwise, we select an open path θ and define

$$\mathsf{v}_0(\mathsf{p}) = \begin{cases} \mathtt{f} & \text{if } F\text{p on } \theta \\ \mathtt{t} & \text{otherwise} \end{cases}$$

Then the proof of the Hintikka lemma shows that S-Value(FX,v_0)=t, thus Value(X,v_0)=f.

Corollary 13: For every formula X it is decidable if there is a tableau proof for X or an interpretation that falsifies it.

James Caldwell has given a formal and constructive proof of Theorem 12 in the Nuprl proof development system. Here, the theorem reads as

Theorem 12: $\forall$X:FORM. $\exists\mathcal{T}$:Tableaux$_X$. closed($\mathcal{T}$) $\vee$ $\exists\mathsf{v}_0$:Var$_X\rightarrow\mathbb{B}$. Value(X,v_0)=f

but both the existential quantifier and the disjunction have a computational meaning, i.e. $\exists x{:}T.P[x]$ *means that there is a program that constructs an* x *with property* $P[x]$ *and* $X \vee Y$ *means that there is an algorithm that decides whether* X *or* Y *is true.*