The Lovász Theta Function and a Semidefinite Programming Relaxation of Vertex Cover

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Abstract

Let $vc(G)$ denote the minimum size of a vertex cover of a graph $G = (V, E)$. It is well-known that one can approximate $vc(G)$ to within a factor of 2 in polynomial time; and despite considerable investigation, no $(2-\varepsilon)$ approximation algorithm has been found for any $\varepsilon > 0$. Because of the many connections between the independence number $\alpha(G)$ and the Lovász theta function $\vartheta(G)$, and because $vc(G) = |V| - \alpha(G)$, it is natural to ask how well $|V| - \vartheta(G)$ approximates $vc(G)$. It is not difficult to show that these quantities are within a factor of 2 of each other ($|V| - \vartheta(G)$ is never less than the value of the canonical linear programming relaxation of $vc(G)$); our main result is that $vc(G)$ can be more than $(2-\varepsilon)$ times $|V| - \vartheta(G)$ for any $\varepsilon > 0$. We also investigate a stronger lower bound than $|V| - \vartheta(G)$ for $vc(G)$.

1 Introduction

Let $G = (V, E)$ be an undirected graph. By a vertex cover of G we mean a set $S \subset V$ such that for each $e \in E$, at least one endpoint of e lies in S. Thus, a vertex cover is the complement of an independent set in G. For a graph in which each vertex i is given a non-negative weight w_i, the problem of finding a vertex cover of minimum total weight is a classical NP-complete problem. We are interested here in the question of finding approximate solutions to this problem in polynomial time.

We can formulate the problem of finding a minimum-weight vertex cover via the following integer program. Assign a variable x_i to each vertex $i \in V$; then we have

$$\begin{array}{lll} \text{(VC)} \quad \text{Min} & \sum_i w_i x_i & \\ \text{s.t.} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in \{0,1\} & i \in V. \end{array}$$

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Let us denote the optimum value of this integer program — i.e. the weight of the optimal vertex cover — by $vc(G)$.

It is well-known that $vc(G)$ can be approximated to within a factor of 2 in polynomial time; one way to see this is as follows. We can relax the constraint that the x_i be 0-1 variables, obtaining the following linear program:

$$\begin{array}{llll} \text{(LP)} & \text{Min} & \sum_i w_i x_i & \\ & \text{s.t.} & x_i + x_j \geq 1 & (i,j) \in E \\ & & 0 \leq x_i \leq 1 & i \in V. \end{array}$$

Let us denote the optimum value of (LP) by $lp(G)$. Then clearly $vc(G) \geq lp(G)$; but we also have $lp(G) \geq vc(G)/2$, as the set

$$\{i : x_i \geq 1/2\}$$

in any feasible solution x to (LP) is easily seen to be a vertex cover for G (Hochbaum [7]). Thus this linear program leads to a 2-approximation algorithm for the vertex cover problem.

There has been considerable work on the problem of finding a polynomial-time approximation algorithm with an improved performance guarantee; the best bound currently known is $2 - \frac{\log \log n}{2 \log n}$ [2, 11]. What is quite striking is that no polynomial-time $(2 - \varepsilon)$-approximation algorithm is known, for any constant $\varepsilon > 0$.

1.1 The present work

In this paper, we consider a number of natural semidefinite programming relaxations of the vertex cover problem, and investigate whether any of these might provide a $(2 - \varepsilon)$-approximation algorithm. Semidefinite programming relaxations have recently proved useful in obtaining improved approximation algorithms for a number of well-studied optimization problems, including maximum cut and satisfiability problems [6], vertex coloring [9], and the maximum independent set problem [1]. Probably the most well-known semidefinite programming relaxation is the *theta function* $\vartheta(G)$ of Lovász [10]. This was introduced as a relaxation of the maximum independent set problem and used in [10] to show the polynomial-time solvability of the maximum independent set and minimum vertex coloring problems in perfect graphs. More recently it has been used in the approximation algorithms of [9] and [1].

Let $\alpha(G)$ denote the maximum weight of an independent set of G, and $W = \sum_{i \in V} w_i$ the sum of all vertex weights in G. Since $vc(G) = W - \alpha(G)$, it is natural to ask how well $W - \vartheta(G)$ approximates $vc(G)$. It is not difficult to show (see Section 2) that $W - \vartheta(G)$ is always at least $lp(G)$, and hence not more than a factor of 2 smaller than $vc(G)$. Our main result is a corresponding lower bound — we construct a family of unweighted graphs for which the ratio of $vc(G)$ to $n - \vartheta(G)$ converges to 2, where $n = |V|$.

The gap between $vc(G)$ and $n - \vartheta(G)$ can also be obtained from a construction due independently to Alon and Kahale [1] — their concern was with the complement of our problem: graphs G with small independence number for which $\vartheta(G)$ converges to $\frac{1}{2}n$. We also note that the recent construction of Feige [4], showing that the ratio

$\vartheta(G)/\alpha(G)$ can be as large as $n^{1-o(1)}$, is of no use for our purposes; for the graphs he deals with, the ratio of $vc(G)$ to $n - \vartheta(G)$ converges to 1, not 2.

In the final section we present a natural strengthening of the formulation; this turns out to be equal to $W - \vartheta'(G)$, where ϑ' denotes the variant of the Lovász theta function introduced by Schrijver [13]. We currently do not know of families of graphs for which the ratio of $W - \vartheta'(G)$ to $vc(G)$ converges to 2, and we indicate how the question of the existence of such examples is closely related to some open problems in combinatorial geometry.

2 The Semidefinite Programming Relaxation

Perhaps the most natural way to obtain our semidefinite programming relaxation is by considering the following quadratic integer programming formulation of $vc(G)$.

$$
\begin{array}{llll}
\text{(VC)} & \text{Min} & \sum_{i\in V} w_i(1+y_0y_i)/2 & \\
 & \text{s.t.} & (y_0-y_i)(y_0-y_j)=0 & (i,j)\in E \\
 & & y_i \in \{-1,+1\} & i \in V \\
 & & y_0 \in \{-1,+1\}, &
\end{array}
$$

where the vertex cover corresponds to the set of vertices i for which $y_i = y_0$. One could of course get rid of y_0 and/or restrict y_i to be in $\{0,1\}$, but this form simplifies the derivation of the relaxation. We now relax this integer program to one in which y_0 and the y_i ($i \in V$) are vectors in $\mathbf{R}^{n+1}$ (where n denotes $|V|$).

$$
\begin{array}{llll}
\text{(SD)} & \text{Min} & \sum_{i\in V} w_i(1+y_0\cdot y_i)/2 & \\
 & \text{s.t.} & (y_0-y_i)\cdot(y_0-y_j)=0 & (i,j)\in E \\
 & & y_i^2 = 1 & i \in V \\
 & & y_0^2 = 1. &
\end{array}
$$

The constraints $(y_0 - y_i)\cdot(y_0 - y_j) = 0$ for $(i,j) \in E$ can also be expressed more geometrically by saying that the midpoint $\frac{1}{2}(y_i + y_j)$ must be on the sphere centered at $y_0/2$ and of radius $1/2$, i.e. that $\left(\frac{y_i+y_j-y_0}{2}\right)^2 = \frac{1}{4}$. The relaxation can be reformulated as a semidefinite program and therefore, using the ellipsoid algorithm, one can determine its optimum to within additive errors in polynomial time. Let us denote the optimum value of this semidefinite program by $sd(G)$. Observe that $sd(G) \le vc(G)$, since for any vertex cover S of G, we obtain a feasible solution to the above semidefinite program as follows. Set y_0 equal to any unit vector u; and for each $i \in V$, set $y_i = y_0$ if $i \in S$ and $y_i = -y_0$ if $i \notin S$.

First let us establish that we are indeed dealing with the theta function.

Theorem 1 *$W - sd(G) = \vartheta(G)$.*

Proof. We can write $W - sd(G)$ as

$$\begin{array}{llll} (\mathrm{SD}^c) & \text{Max} & \sum_{i \in V} w_i(1 - y_0 \cdot y_i)/2 & \\ & \text{s.t.} & (y_0 - y_i) \cdot (y_0 - y_j) = 0 & (i,j) \in E \\ & & y_i^2 = 1 & i \in V \\ & & y_0^2 = 1. & \end{array}$$

We use the following formulation of the theta function [10]; there is a unit vector $u_i \in \mathbf{R}^{n+1}$ for each vertex of G, and an additional unit vector $d \in \mathbf{R}^{n+1}$.

$$\begin{array}{llll} (\vartheta) & \text{Max} & \sum_{i \in V} w_i(d \cdot u_i)^2 & \\ & \text{s.t.} & u_i \cdot u_j = 0 & (i,j) \in E \\ & & u_i^2 = 1 & i \in V \\ & & d^2 = 1. & \end{array}$$

We claim first that $W - sd(G) \leq \vartheta(G)$. Given a feasible solution to (SD^c), set $d = y_0$; for each $i \in V$, we set

$$u_i = \frac{y_0 - y_i}{\|y_0 - y_i\|}$$

if $y_0 \neq y_i$, and otherwise we choose u_i to be any unit vector orthogonal to d and all other u_j. For this set of unit vectors, we have $u_i \cdot u_j = 0$ for $(i,j) \in E$. We compute the value of the objective function as follows. If $y_0 \neq y_i$, then

$$\begin{aligned} (d \cdot u_i)^2 &= \frac{[y_0 \cdot (y_0 - y_i)]^2}{(y_0 - y_i)^2} \\ &= \frac{(1 - y_0 \cdot y_i)^2}{2(1 - y_0 \cdot y_i)} \\ &= \frac{1}{2}(1 - y_0 \cdot y_i). \end{aligned}$$

If $y_0 = y_i$ then

$$(d \cdot u_i)^2 = 0 = \frac{1}{2}(1 - y_0 \cdot y_i).$$

As a result, we have constructed a feasible solution to (ϑ) of value $W - sd(G)$.

Conversely, we show that $\vartheta(G) \leq W - sd(G)$. Given a feasible solution to (ϑ), write $y_0 = d$ and $y_i = d - 2(d \cdot u_i)u_i$. Then $y_i^2 = 1$, and if $(i,j) \in E$ we have

$$(y_0 - y_i) \cdot (y_0 - y_j) = 4(d \cdot u_i)(d \cdot u_j)(u_i \cdot u_j) = 0.$$

Finally,

$$\frac{1}{2}(1 - y_0 \cdot y_i) = \frac{1}{2}(2(d \cdot u_i)^2) = (d \cdot u_i)^2.$$

∎

The next two results determine the exact approximation ratio achieved by $sd(G)$ — specifically, $vc(G) \leq 2sd(G)$ but, for any $\varepsilon > 0$, there exist instances for which $vc(G) > (2 - \varepsilon)sd(G)$. It is worth noting, however, that on many natural examples, $sd(G)$ is a much tighter relaxation than $lp(G)$. For instance on K^n, the complete graph on n vertices with unit weights, one has $lp(G) = \frac{1}{2}n$, while $sd(G) = vc(G) = n - 1$.

Proposition 1 *$sd(G) \geq lp(G)$.*

Proof. Suppose we have a feasible solution to (SD), and we write $x_i = (1 + y_0 \cdot y_i)/2$. Then we claim that $\{x_i : i \in V\}$ is a feasible solution to (LP). For clearly $0 \leq x_i \leq 1$; and if $(i,j) \in E$, then $(y_0 - y_i) \cdot (y_0 - y_j) = 0$, whence $y_0 \cdot y_i + y_0 \cdot y_j = 1 + y_i \cdot y_j$ and

$$x_i + x_j = \frac{3}{2} + \frac{1}{2} y_i \cdot y_j \geq 1$$

as required. ∎

Theorem 2 *For each $\varepsilon > 0$ there is a graph G_ε on $n = n(\varepsilon)$ vertices, with all vertex weights equal to 1, for which $vc(G_\varepsilon)/sd(G_\varepsilon) \geq 2 - \varepsilon$.*

Proof. For a point $x \in \mathbf{R}^d$, let $x^{(i)}$ denote the i^{th} coordinate of x. Also, let $e_1, \ldots, e_d$ denote the coordinate unit vectors in $\mathbf{R}^d$.

The idea is to construct a graph G_ε as follows. The vertices of G_ε will be the set of all $n = 2^m$ many m-bit strings of zeroes and ones, for some sufficiently large value of m, and two vertices will be joined by an edge if their Hamming distance is equal to $(1-\gamma)m$, for some small $\gamma > 0$ depending on ε. Thus, two vertices will be joined if they are nearly antipodal under the Hamming metric. We then obtain a solution to (SD) in which all y_i $(i \in V)$ are nearly orthogonal to y_0, by mapping the y_i to the vertices of an "inscribed" hypercube in a copy of the m-dimensional unit ball. Thus $sd(G_\varepsilon)$ is close to $n/2$. But using a theorem of Frankl and Rödl [5], we can show that G_ε does not have large independent sets, and thus that $vc(G_\varepsilon)$ is close to n.

The details are as follows. Let ε' be a rational number such that $\varepsilon' \leq \varepsilon$. Let

$$\alpha = \frac{\varepsilon'}{4}$$

$$\beta = \sqrt{1 - \alpha^2}$$

$$\gamma = \frac{1}{2} - \frac{(1-\alpha)^2}{2\beta^2}.$$

Note that $\gamma > 0$. The vertex set of G_ε consists of all m-bit strings of zeroes and ones, where the value of m will be determined below; for now, we only require that $(1-\gamma)m$ be an even integer. If i and j are vertices of G_ε, then $(i,j) \in E$ iff the Hamming distance between i and j is equal to $(1-\gamma)m$.

First we compute an upper bound on $sd(G_\varepsilon)$. To do this, we construct the following unit vectors in $\mathbf{R}^{m+1}$. Set $y_0 = e_{m+1}$. For $i \in V$, define y_i so that $y_i^{(p)} = \beta/\sqrt{m}$ if the p^{th} bit of i is 1, and $y_i^{(p)} = -\beta/\sqrt{m}$ if it is 0. Finally, $y_i^{(m+1)} = \alpha$ for all $i \in V$; thus all y_i are unit vectors.

Now, if $(i,j) \in E$, then i and j have Hamming distance $(1-\gamma)m$, and hence

$$\begin{aligned}
(y_0 - y_i) \cdot (y_0 - y_j) &= (1-\alpha)^2 + \gamma m(\beta^2/m) - (1-\gamma)m(\beta^2/m) \\
&= (1-\alpha)^2 - \beta^2(1-2\gamma) \\
&= 0
\end{aligned}$$

by the definition of γ. Thus the given vectors constitute a feasible solution for (SD). Moreover, the value of the objective function with these vectors is equal to $\frac{1}{2}(1+\alpha)n$, so

$$sd(G_\varepsilon) \leq \frac{1}{2}(1+\alpha)n.$$

Now we show a lower bound on $vc(G_\varepsilon)$; for this we need the following theorem of Frankl and Rödl [5].

> Let $\mathcal{C}$ be a collection of m-bit strings, ξ a constant satisfying $0 < \xi < \frac{1}{2}$, and d an even integer satisfying $\xi m < d < (1-\xi)m$. Then for some constant δ depending only on ξ, if $|\mathcal{C}| > (2-\delta)^m$, then $\mathcal{C}$ contains two strings with Hamming distance exactly d.

For our purposes, choose $\xi < \gamma$ and let δ denote the constant obtained by applying this theorem. Now, let $d = (1-\gamma)m$, where m is chosen large enough so that d is an even integer and

$$(2-\delta)^m \leq \alpha \cdot 2^m.$$

Thus, in G_ε any set of more than $\alpha \cdot 2^m = \alpha n$ vertices contains the two endpoints of some edge, so the largest independent set in G_ε has size at most αn. Since the complement of any vertex cover is an independent set, this implies

$$vc(G_\varepsilon) \geq (1-\alpha)n.$$

The theorem now follows, since

$$\begin{aligned} \frac{vc(G_\varepsilon)}{sd(G_\varepsilon)} &\geq \frac{(1-\alpha)n}{\frac{1}{2}(1+\alpha)n} \\ &\geq 2-\varepsilon. \end{aligned}$$

∎

3 Strengthening the Relaxation

It turns out that we can add a set of very natural valid inequalities to (SD) that rules out the bad example of the previous section. As we remarked in the introduction, this new formulation (SD$'$) is in fact equal to $W - \vartheta'(G)$, where ϑ' denotes the variant of the Lovász theta function introduced by Schrijver [13].

The new formulation is obtained by observing the following. We saw that for any vertex cover S, we can obtain a feasible solution to (SD) by setting $y_i = y_0$ for $i \in S$ and $y_i = -y_0$ for $i \notin S$. But such a solution satisfies the conditions $(y_0 - y_i)\cdot(y_0 - y_j) \geq 0$ *for all* pairs of vertices $i, j \in V$, regardless of whether $(i,j) \in E$. Thus we can write

$$\begin{array}{llll} (\mathrm{SD}') & \mathrm{Min} & \sum_{i\in V} w_i(1 + y_0 \cdot y_i)/2 & \\ & \mathrm{s.t.} & (y_0 - y_i)\cdot(y_0 - y_j) = 0 & (i,j) \in E \\ & & (y_0 - y_i)\cdot(y_0 - y_j) \geq 0 & \forall\ i,j \in V \\ & & y_i^2 = 1 & i \in V \\ & & y_0^2 = 1. & \end{array}$$

Let us denote the optimum value of (SD$'$) by $sd'(G)$.

The function $\vartheta'(G)$ was introduced by Schrijver [13]. As in the definition of ϑ, we have a unit vector $u_i \in \mathbf{R}^{n+1}$ for each vertex of G, and an additional unit vector $d \in \mathbf{R}^{n+1}$. We can now formulate $\vartheta'(G)$ as follows.

$$
\begin{array}{llll}
(\vartheta') & \text{Max} & \sum_{i \in V} w_i (d \cdot u_i)^2 & \\
& \text{s.t.} & u_i \cdot u_j = 0 & (i,j) \in E \\
& & u_i \cdot u_j \geq 0 & \forall\ i,j \in V \\
& & d \cdot u_i \geq 0 & i \in V \\
& & u_i^2 = 1 & i \in V \\
& & d^2 = 1. &
\end{array}
$$

By a straightforward modification of the proof of Theorem 1, we have

Theorem 3 $sd'(G) = W - \vartheta'(G)$.

Now it is easy to verify that the set of vectors we constructed in the proof of Theorem 2 is no longer feasible for (SD$'$). But in fact we can say more. Let $U = \{u_1, \ldots, u_n\}$ denote a set of points in $\mathbf{R}^d$, and define d_U by

$$d_U = \max_{u_i, u_j \in U} \|u_i - u_j\|.$$

We now associate a graph $\mathcal{K}_U$ with U as follows. $\mathcal{K}_U$ contains a vertex i for each $u_i \in U$; we join i and j by an edge iff $\|u_i - u_j\| = d_U$.

Graphs of the form $\mathcal{K}_U$ are of considerable interest in combinatorial geometry because of their role in the well-known Borsuk Conjecture [3], which asked (in its finite form) whether $\chi(\mathcal{K}_U) \leq d+1$ for all point sets U in $\mathbf{R}^d$. (This is the bound achieved for example by the unit d-simplex.) This was recently answered negatively by Kahn and Kalai [8], who constructed, for infinitely many values of d, a set U in $\mathbf{R}^d$ for which $\chi(\mathcal{K}_U) \geq (1.2)^{\sqrt{d}}$.

Here we ask a related question. Let S^{d-1} denote the unit sphere centered at the origin in $\mathbf{R}^d$.

Question 1 *Do there exist constants $\varepsilon > 0$ and $\delta > 0$ so that for all sets U of n points on S^{d-1}, $d_U \geq 2 - \varepsilon$ implies $\alpha(\mathcal{K}_U) \geq \delta n$?*

That is, does every point set of sufficiently large diameter on S^{d-1} have a linear-sized independent set in its graph $\mathcal{K}_U$? It is important to note that the constants ε and δ do not depend on n or d.

The relation of this to our formulation (SD$'$) is contained in the following fact.

Proposition 2 *If for some $c < 2$ we have $vc(G)/sd'(G) < c$ for all graphs G, then Question 1 has an affirmative answer.*

Proof. Given $c < 2$, let $\varepsilon = 1 - \frac{c}{2} > 0$ and $\delta = \varepsilon^2/4 > 0$. Consider a set $U = \{u_1, \ldots, u_n\}$ on S^{d-1} for which $d_U \geq 2 - \varepsilon$.

We first claim that $sd'(\mathcal{K}_U) \le \frac{2}{(2-\varepsilon)^2}n$. Select any unit vector y_0 orthogonal to all u_i's (adding one dimension if necessary). Let $y_i = \beta y_0 + \sqrt{1-\beta^2}u_i$, where $\beta = \frac{4}{d_U^2} - 1$. Observe that y_i is a unit vector and that

$$\begin{aligned}(y_0 - y_i)\cdot(y_0 - y_j) &= ((1-\beta)y_0 - \sqrt{1-\beta^2}u_i)\cdot((1-\beta)y_0 - \sqrt{1-\beta^2}u_j) \\ &= (1-\beta)^2 + (1-\beta^2)(u_i \cdot u_j) \\ &= (1-\beta)(1-\beta+(1+\beta)u_i\cdot u_j).\end{aligned}$$

Since the u_i are unit vectors, $||u_i - u_j||^2 = 2 - 2u_i\cdot u_j$. Substituting this into (1), we derive that

$$\begin{aligned}(y_0 - y_i)\cdot(y_0 - y_j) &= (1-\beta)\left(2 - \frac{1+\beta}{2}||u_i - u_j||^2\right) \\ &\ge (1-\beta)(2 - \frac{1+\beta}{2}d_U^2) \\ &= 0,\end{aligned}$$

with equality if $||u_i - u_j|| = d_U$. We have therefore constructed a feasible solution to (SD') of value $\frac{1+\beta}{2}n = \frac{2}{d_U^2}n \le \frac{2}{(2-\varepsilon)^2}n$.

From the assumption, we know that $vc(\mathcal{K}_U) < c \cdot sd'(\mathcal{K}_U)$. Thus,

$$vc(\mathcal{K}_U) < c\frac{2}{(2-\varepsilon)^2}n = \frac{4-4\varepsilon}{4-4\varepsilon+\varepsilon^2}n < \left(1 - \frac{\varepsilon^2}{4}\right)n = (1-\delta)n,$$

implying that $\alpha(\mathcal{K}_U) > \delta n$. ∎

We do not know the answer to Question 1, but it is worth remarking on its relation to a number of other questions.

The Counterexample to Borsuk's Conjecture. In their counterexample to Borsuk's Conjecture, Kahn and Kalai construct a family of sets $\{U_n\}$ such that U_n is a set of n points on a unit sphere, and $\alpha(\mathcal{K}_U) = o(n)$. However, their sets U_n also have $d_{U_n} = \sqrt{2} + o(1)$. Thus, following the proof of Proposition 2, their construction is not sufficient to exhibit a family of graphs $\{G_n\}$ for which $vc(G_n)/sd'(G_n) \ge c'$, for *any* constant $c' > 1$.

Strongly Self-Dual Polytopes. Lovász in [12] introduces the notion of a *strongly self-dual polytope*, which is defined as a polytope P in $\mathbf{R}^d$ with the following properties.

(i) The vertices of P all lie on S^{d-1}.

(ii) For some $0 < r < 1$, P is circumscribed around the sphere of radius r centered at the origin.

(iii) There is a bijection between the vertices and facets of P so that the vector from the origin to any vertex of P is orthogonal to the corresponding facet.

Note that in $\mathbf{R}^2$, an odd regular polygon satisfies these conditions. Let $U(P)$ denote the vertices of P; Lovász proves the following two facts.

- For each dimension d, and all $\varepsilon > 0$, there is a strongly-self dual polytope P in $\mathbf{R}^d$ with $d_{U(P)} \ge 2 - \varepsilon$.

- For any strongly self-dual polytope P in $\mathbf{R}^d$, $\chi(\mathcal{K}_{U(P)}) \geq d+1$.

Taken together, these two facts provide a negative answer to the following — a slight modification of a question due to Erdös and Graham.

Question 2 *Do there exist constants $\varepsilon > 0$ and $C > 0$ so that for all sets U of n points on S^{d-1}, $d_U \geq 2-\varepsilon$ implies $\chi(\mathcal{K}_U) \leq C$?*

Clearly an affirmative answer to Question 2 would have implied an affirmative answer to Question 1. On the other hand, an affirmative answer to Question 1 implies that if $d_U \geq 2-\varepsilon$ then $\chi(\mathcal{K}_U) \leq 1+\log_{(1/(1-\delta))} n = O(\log n)$. This simply corresponds to repeatedly coloring a fraction δ of the vertex set with a new color. For this argument, we also need to observe that we could apply the affirmative answer to Question 1 to an induced subgraph of $\mathcal{K}_U$ (unless it is itself an independent set).

However, looking at Lovász's construction, one finds that for the strongly self-dual polytopes P he constructs, one always has $\alpha(\mathcal{K}_{U(P)}) \geq \frac{n-1}{2}$ (with equality achieved, for example, on all regular odd polygons in $\mathbf{R}^2$). Thus his construction is not able to provide a negative answer to Question 1; as with [8], it does not provide a family of graphs $\{G_n\}$ for which $vc(G_n)/sd'(G_n) \geq c'$, for any constant $c' > 1$.

Naturally, it would be interesting to investigate other constructions of strongly self-dual polytopes.

A Graph-Coloring Formulation. Consider the following special case of Question 1, to which we also do not know the answer.

Question 3 *Can one take some $\varepsilon > 2-\sqrt{3}$ in Question 1?*

Then we have

Proposition 3 *If Question 3 has an affirmative answer, then for some $C > 0$ one can prove in polynomial time that a graph of chromatic number at least $C\log n$ is not 3-colorable.*

Proof. Given a graph $G=(V,E)$, consider the problem of finding unit vectors u_i's such that E is a subset of the edge set of $\mathcal{K}_U$ and such that d_U is maximized. Let d_{max} be the maximum achievable. Since $||u_i-u_j||^2 = 2-2u_i\cdot u_j$, this problem can be formulated in terms of a semidefinite program:

$$\begin{array}{lll} \text{Min} & z & \\ \text{s.t.} & u_i \cdot u_j = z & (i,j) \in E \\ & u_i \cdot u_j \geq z & (i,j) \notin E \\ & u_i^2 = 1 & i \in V. \end{array}$$

Now suppose that Question 1 has an affirmative answer for *some* constants $\varepsilon > 0$ and $\delta > 0$. By the remark following Question 2, we know that if the chromatic number of G is at least $C\log n$ (for some C depending on δ) then there are no vectors u_i with the desired properties for which $d_U \geq 2-\varepsilon$. But, in polynomial time, we can determine d_{max} to within any additive error by solving the above semidefinite program.

In particular, if $\varepsilon > 2 - \sqrt{3}$, we can prove in polynomial time that $d_{max} < \sqrt{3}$. This constitutes a proof that the graph is not 3-colorable, since a 3-coloring would imply $d_{max} \geq \sqrt{3}$ (as in [9]). ■

References

[1] N. Alon and N. Kahale, "Approximating the independence number via the θ-function," manuscript.

[2] R. Bar-Yehuda and S. Even, "A local-ratio theorem for approximating the weighted vertex cover problem", in *Analysis and Design of Algorithms for Combinatorial Problems*, Amsterdam: Elsevier, 1985, pp. 27-46.

[3] K. Borsuk, "Drei Sätze über die n-dimensionale euklidische Sphäre," *Fundamenta Math.*, 20(1933), pp. 177–190.

[4] U. Feige, "Randomized graph products, chromatic numbers, and the Lovász θ-function," *Proc. 27th ACM Symposium on Theory of Computing*, 1995.

[5] P. Frankl and V. Rödl, "Forbidden intersections," *Transactions of the American Mathematical Society*, 300(1987), pp. 259–286.

[6] M. Goemans and D. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," to appear in *Journal of the ACM.* (Preliminary version in *Proc. 26th ACM Symposium on Theory of Computing*, 1994.)

[7] D. Hochbaum, "Approximation algorithms for set covering and vertex cover problems," *SIAM J. Computing*, 11(1982), pp. 555–556.

[8] J. Kahn and G. Kalai, "A counterexample to Borsuk's Conjecture," *Bulletin of the American Mathematical Society*, 29(1993), pp. 60–62.

[9] D. Karger, R. Motwani and M. Sudan, "Approximate graph coloring by semidefinite programming," *Proc. 35th IEEE Symposium on Foundations of Computer Science*, 1994, pp. 2–13.

[10] L. Lovász, "On the Shannon capcity of a graph," *IEEE Transactions on Information Theory*, 25(1979), pp. 1–7.

[11] B. Monien and E. Speckenmeyer, "Ramsey numbers and an approximation algorithm for the vertex cover problem", *Acta Informatica* 22(1985), pp. 115–123.

[12] L. Lovász, "Self-dual polytopes and the chromatic number of distance graphs on the sphere," *Acta Sci. Math.*, 45(1983), pp. 317–323.

[13] A. Schrijver, "A comparison of the Delsarte and Lovász bounds," *IEEE Transactions on Information Theory*, 25(1979), pp. 425–429.