Reasoning About Common Knowledge with Infinitely Many Agents

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Abstract

Complete axiomatizations and exponential-time decision procedures are provided for reasoning about knowledge and common knowledge when there are infinitely many agents. The results show that reasoning about knowledge and common knowledge with infinitely many agents is no harder than when there are finitely many agents, provided that we can check the cardinality of certain set differences $G - G'$, where G and G' are sets of agents. Since our complexity results are independent of the cardinality of the sets G involved, they represent improvements over the previous results even with the sets of agents involved are finite. Moreover, our results make clear the extent to which issues of complexity and completeness depend on how the sets of agents involved are represented.

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1 Introduction

Reasoning about knowledge and common knowledge has been shown to be widely applicable in distributed computing, AI, and game theory. (See [FHMV95] for numerous examples.) Complete axioms for reasoning about knowledge and common knowledge are well known in the case of a fixed finite set of agents. However, in many applications, the set of agents is not known in advance and has no *a priori* upper bound (think of software agents on the web or nodes on the Internet, for example); it is often easiest to model the set of agents as an infinite set. Infinite sets of agents also arise in game theory and economics (where reasoning about knowledge and common knowledge is quite standard; see, for example, [Aum76, Gea94]). For example, when analyzing a game played with two teams, we may well want to say that everyone on team 1 knows that everyone on team 2 knows some fact p, or that it is common knowledge among the agents on team 1 that p is common knowledge among the agents on team 2. We would want to say this even if the teams consist of infinitely many agents. Since economies are often modeled as consisting of infinitely many (even uncountably many) agents, this type of situation arises when economies are viewed as teams in a game.

The logics for reasoning about the knowledge of groups of agents contain modal operators K_i (where $K_i\varphi$ is read "agent i knows φ"), E_G (where $E_G\varphi$ is read "everyone in group G knows φ"), and C_G (where $C_G\varphi$ is read "φ is common knowledge among group G"). The operators E_G and C_G make perfect sense even if we allow the sets G to be infinite—their semantic definitions remain unchanged. If the set of agents is finite, so that, in particular, G is finite, there is a simple axiom connecting $E_G\varphi$ to $K_i\varphi$, namely, $E_G\varphi \Leftrightarrow \wedge_{i\in G}K_i\varphi$. Once we allow infinite groups G of agents, there is no obvious analogue for this axiom. Nevertheless, in this paper, we show that there exist natural sound and complete axiomatizations for reasoning about knowledge and common knowledge even if there are infinitely many agents.

It is also well known that if there are finitely many agents, then there is a decision procedure that decides if a formula φ is satisfiable (or valid) that runs in time exponential in $|\varphi|$, where φ is the length of the formula viewed as a string of symbols. We prove a similar result for a language with infinitely many agents. However, two issues arise (that, in fact, are also relevant even if there are only finitely many agents, although they have not been considered before):

- In the statement of the complexity result in [FHMV95], E_G and C_G are both viewed as having length $2 + 2|G|$ (where $|G|$ is the cardinality of G). Clearly we cannot use this definition here if we want to get interesting complexity results, since $|G|$ may be infinite. Even if we restrict our attention to finite sets G, we would like a decision procedure that treats these sets in a uniform way, independently of their cardinality. Here we view E_G as having length 1 and C_G as having length 3, independent of the cardinality of G. (See, for example, the proof of Proposition 3.5 for the role of independence and the definition of $Sub(\varphi)$ in the proof of Theorem 4.5 for an indication as to why C_G has length 3 rather than 1.) Even with this definition of length, we prove that the complexity of the satisfiability problem is still essentially exponential time. (We discuss below what "essentially" means.) Thus our results improve previously-known results even if there are only finitely many agents.
- In the earlier proofs, it is implicitly assumed that the sets G are presented in such a way that there is no difficulty in testing membership in G. As we show here, in order to decide

if certain formulas are satisfiable, we need to be able to test if certain subsets of agents of the form $G_0 - (G_1 \cup \ldots \cup G_k)$ are empty, where $G_0, \ldots, G_k$ are sets of agents. In fact, if we are interested in a notion of knowledge that satisfies *positive introspection*—that is, if agent i knows φ, then she knows that she knows it—then we also must be able to check whether such subsets are singletons. And if we are interested in a notion of knowledge that satisfies *negative introspection*—that is, if agent i does not know φ, then she knows that she does not know it—then we must be able to check whether such subsets have cardinality m, for certain finite m. The difficulty of deciding these questions depends in part on how $G_0, \ldots, G_k$ are presented and which sets of agents we can talk about in the language. For example, if $G_0, \ldots, G_k$ are recursive sets, deciding if $G_0 - (G_1 \cup \ldots \cup G_k)$ is nonempty may not even be recursive. Here, we provide a decision procedure for satisfiability that runs in time exponential in $|\varphi|$ provided that we have oracles for testing appropriate properties of sets of the form $G_0 - (G_1 \cup \ldots \cup G_k)$. Moreover, we show that any decision procedure must be able to answer the questions we ask. In fact, we actually prove a stronger result, providing a tight bound on the complexity of deciding satisfiability that takes into account the complexity of answering questions about the cardinality of $G_0 - (G_1 \cup \ldots \cup G_k)$.

Again, this issue is of significance even if there are only finitely many agents. For example, in the SDSI approach to security [RL96], there are names, which can be viewed as representing sets of agents. SDSI provides a (nondeterministic) algorithm for computing the set of agents represented by a name. If we want to make statements such as "every agent represented by name `n` knows φ" (statements that we believe will be useful in reasoning about security [HvdM99, HvdMS99]) then the results of this paper show that to decide validity in the resulting logic, we need more than just an algorithm for resolving the agents represented by a given name. We also need algorithms for resolving which agents are represented by one name and not another. More generally, if we assume that we have a separate language for representing sets of agents, our results characterize the properties of sets that we need to be able to decide in order to reason about the group knowledge of these agents.

In the next section, we briefly review the syntax and semantics of the logic of common knowledge. In Section 3 we state the main results and prove them under some simplifying assumptions that allow us to bring out the main ideas of the proof. We drop these assumptions in Section 4, where we provide the proofs of the full results.

2 Syntax and Semantics: A Brief Review

Syntax: We start with a (possibly infinite) set $\mathcal{A}$ of agents. Let $\mathcal{G}$ be a set of nonempty subsets of $\mathcal{A}$. (Note that we do not require $\mathcal{G}$ to be closed under union, intersection, or complementation; it can be an arbitrary collection of subsets.) We get the language $\mathcal{L}^C_{\mathcal{G}}(\Phi)$ by starting with a set Φ of primitive propositions, and closing under $\wedge$, $\neg$, and the modal operators K_i, for $i \in \mathcal{A}$, and E_G, C_G, for $G \in \mathcal{G}$. Thus, if $p, q \in \Phi$, $i \in \mathcal{A}$, and $G, G' \in \mathcal{G}$, then $K_i C_G(p \wedge E_{G'} q) \in \mathcal{L}^C_{\mathcal{G}}(\Phi)$. Let $\mathcal{L}^E_{\mathcal{G}}$ be the sublanguage of $\mathcal{L}^C_{\mathcal{G}}$ that does not include the C_G operators. Let $|\varphi|$ be the length of the formula viewed as a string of symbols, where the modal operators K_i and E_G are counted as having length 1 and C_G is counted as having length 3 (even if G is an infinite set of agents) and all primitive propositions are counted as having length 1.

In [FHMV95, HM92], $\mathcal{A}$ is taken to be the set $\{1, \ldots, n\}$; in [HM92], $\mathcal{G}$ is taken to be the singleton $\{\{1, \ldots, n\}\}$ (so that we can only talk about every agent in $\mathcal{A}$ knowing φ and common knowledge among the agents in $\mathcal{A}$), while in [FHMV95], $\mathcal{G}$ is taken to consist of all nonempty subsets of $\mathcal{A}$.

Semantics: As usual, formulas in $\mathcal{L}_{\mathcal{G}}^{C}$ are either true or false at a world in a Kripke structure. Formally, a Kripke structure M over $\mathcal{A}$ and Φ is a tuple $(S, \pi, \{\mathcal{K}_i : i \in \mathcal{A}\})$, where S is a set of states or possible worlds, π associates with each state in S a truth assignment to the primitive propositions in Φ (so that $\pi(s) : \Phi \to \{\mathbf{true}, \mathbf{false}\}$), and $\mathcal{K}_i$ is a binary relation on S for each agent $i \in \mathcal{A}$. We occasionally write $\mathcal{K}_i(s)$ for $\{t : (s,t) \in \mathcal{K}_i\}$.

We define the truth relation $\models$ as follows:

$(M, s) \models p$ (for $p \in \Phi$) iff $\pi(s)(p) = \mathbf{true}$

$(M, s) \models \varphi \wedge \psi$ iff both $(M, s) \models \varphi$ and $(M, s) \models \psi$

$(M, s) \models \neg\varphi$ iff $(M, s) \not\models \varphi$

$(M, s) \models K_i\varphi$ iff $(M, t) \models \varphi$ for all $t \in \mathcal{K}_i(s)$

$(M, s) \models E_G\varphi$ iff $(M, s) \models K_i\varphi$ for all $i \in G$

$(M, s) \models C_G\varphi$ iff $(M, s) \models E_G^k\varphi$ for $k = 1, 2, 3, \ldots$, where E_G^k is defined inductively by taking $E_G^1\varphi =_{\rm def} E_G\varphi$ and $E_G^{k+1}\varphi =_{\rm def} E_G E_G^k\varphi$.

We say that t is *G-reachable from s in M* if there exist $s_0, \ldots, s_k$ with $s = s_0$, $t = s_k$, and $(s_i, s_{i+1}) \in \cup_{i \in G}\mathcal{K}_i$. For later use, we extend this definition so that if $S' \subseteq S$, we say that t is *G-reachable from s in S'* if $s_0, \ldots, s_k \in S'$. The following characterization of common knowledge is well known [FHMV95].

Lemma 2.1: *$(M, s) \models C_G\varphi$ iff $(M, t) \models \varphi$ for all t that are G-reachable from s in M.*

Let $\mathcal{M}_{\mathcal{A}}(\Phi)$ be the class of all Kripke structures over $\mathcal{A}$ and Φ (with no restrictions on the $\mathcal{K}_i$ relations). We are also interested in various subclasses of $\mathcal{M}_{\mathcal{A}}(\Phi)$, obtained by restricting the $\mathcal{K}_i$ relations. In particular, we consider $\mathcal{M}_{\mathcal{A}}^{r}(\Phi)$, $\mathcal{M}_{\mathcal{A}}^{rt}(\Phi)$, $\mathcal{M}_{\mathcal{A}}^{rst}(\Phi)$, and $\mathcal{M}_{\mathcal{A}}^{elt}(\Phi)$, the class of all structures over $\mathcal{A}$ and Φ where the $\mathcal{K}_i$ relations are reflexive (resp., reflexive and transitive; reflexive, symmetric, and transitive; Euclidean,[1] serial, and transitive). For the remainder of this paper, we take Φ to be fixed, and do not mention it, writing, for example $\mathcal{L}_{\mathcal{G}}^{C}$ and $\mathcal{M}_{\mathcal{A}}$ rather than $\mathcal{L}_{\mathcal{G}}^{C}(\Phi)$ and $\mathcal{M}_{\mathcal{A}}(\Phi)$.

As usual, we define a formula to be *valid in a class* $\mathcal{M}$ of structures if $(M, s) \models \varphi$ for all $M \in \mathcal{M}$ and all states s in M; similarly, φ is *satisfiable in* $\mathcal{M}$ if $(M, s) \models \varphi$ for some $M \in \mathcal{M}$ and some s in M.

[1] Recall that a relation R is Euclidean if $(s,t), (s,u) \in R$ implies that $(t,u) \in R$.

Axioms: The following are the standard axioms and rules that have been considered for knowledge; They hold for all $i \in \mathcal{A}$.

Prop. All substitution instances of tautologies of propositional calculus.

K1. $(K_i\varphi \wedge K_i(\varphi \Rightarrow \psi)) \Rightarrow K_i\psi$.

K2. $K_i\varphi \Rightarrow \varphi$.

K3. $\neg K_i \mathit{false}$.

K4. $K_i\varphi \Rightarrow K_iK_i\varphi$.

K5. $\neg K_i\varphi \Rightarrow K_i\neg K_i\varphi$.

MP. From φ and $\varphi \Rightarrow \psi$ infer ψ.

KGen. From φ infer $K_i\varphi$.

Technically, Prop and K1–K5 are axiom schemes, rather than single axioms. K1, for example, holds for all formulas φ and ψ. A formula such as $K_1q \vee \neg K_1q$ is an instance of axiom Prop (since it is a substitution instance of the propositional tautology $p \vee \neg p$, obtained by substituting K_1q for p).

We will be interested in the following axioms and rule for reasoning about everyone knows, which hold for all $G \in \mathcal{G}$.

E1. $E_G\varphi \Rightarrow K_i\varphi$ if $i \in G$.

E2. $(\wedge_{i \in \mathcal{A}'} K_i\varphi \wedge \wedge_{G' \in \mathcal{G}'} E_{G'}\varphi) \Rightarrow E_G\varphi$ if $\mathcal{A}'$ is a finite subset of $\mathcal{A}$, $\mathcal{G}'$ is a finite subset of $\mathcal{G}$, and $G \subseteq (\mathcal{A}' \cup (\cup\mathcal{G}'))$.

E3. $(E_G\varphi \wedge E_G(\varphi \Rightarrow \psi)) \Rightarrow E_G\psi$.

E4. $E_G(E_G\varphi \Rightarrow \varphi)$.

E5. $E_G\varphi \Rightarrow \varphi$.

E6. $\neg\varphi \Rightarrow E_G\neg E_G\varphi$.

E7. From $\neg(\varphi_1 \wedge \ldots \wedge \varphi_k)$ infer $\neg(E_{G_1}\varphi_1 \wedge \ldots \wedge E_{G_k}\varphi_k)$ if $G_1 \cap \ldots \cap G_k \neq \emptyset$.

EGen. From φ infer $E_G\varphi$.

E2 can be viewed as a generalization of the axiom $E_G\varphi \Rightarrow E_{G'}\varphi$ if $G' \subseteq G$ (of which E1 is a special case if we identify $K_i\varphi$ with $E_{\{i\}}\varphi$, as we often do in the paper). Essentially it says that if $K_i\varphi$ hold for all agents $i \in G$ (and perhaps some other agents $i \notin G$) then $E_G\varphi$ holds. Since, if G is infinite, we cannot write the infinite conjunction of $K_i\varphi$ for all $i \in G$, we approximate as well as we can within the constraints of the language. As long as $E_{G'}\varphi$ and $K_i\varphi$ holds for sets G' and agents i whose union contains G, then certainly $E_G\varphi$ holds.

If $\mathcal{A}$ is finite (so that all the sets in $\mathcal{G}$ are finite) we can simplify E1 and E2 to

E. $E_G\varphi \Leftrightarrow \wedge_{i\in G}K_i\varphi$.

It is easy to see that E follows from E1 and E2 (in the presence of Prop and MP) and every instance of E1 and E2 follows from E if $\mathcal{A}$ is finite. E is used instead of E1 and E2 in [FHMV95, HM92]. Note that E2 is recursive iff deciding if $G-(\mathcal{A}'\cup(\cup\mathcal{G}'))=\emptyset$ is recursive. (We determine precisely which such questions we must be able to answer in Proposition 3.3.)

E3 and EGen are the obvious analogues of K1 and KGen for E_G. We do not need them in the case that $\mathcal{A}$ is finite; it is easy to see that they follow from K1, KGen, and E. In the case that $\mathcal{A}$ is infinite, however, they are necessary.

Axiom E4 is sound in $\mathcal{M}^r_{\mathcal{A}}$, $\mathcal{M}^{rt}_{\mathcal{A}}$, $\mathcal{M}^{rst}_{\mathcal{A}}$, and $\mathcal{M}^{elt}_{\mathcal{A}}$. It is easy to see that E4 follows from K2, E1, and EGen, so will not be needed in systems that contain these axioms. Moreover, it is not hard to show that E4 follows from E1, E2, and K5 if the set of agents is finite. However, it does not follow from these axioms if the set of agents is infinite.

Axiom E5 follows from K2 and E1. Moreover, we use it only in systems that already include K2 and E1. Nevertheless, for technical reasons, it is useful to list it separately. Similarly, it is not hard to see that E7 is a derivable rule in any system that includes Prop, MP, K1, K3, E1, E4, and EGen (we prove this in Section 4.4). While we use E7 only in such systems, like E5, it is useful to list it separately.

Axiom E6 (with E_G replaced by K_i) is the standard axiom used to characterize symmetric $\mathcal{K}_i$ relations [FHMV95]. It follows easily from K2, K5, E1, and E2 if $\mathcal{A}$ is finite. However, like E4, it must be specifically included if $\mathcal{A}$ is infinite.

Finally, we have the following well-known axiom and inference rule for common knowledge:

C1. $C_G\varphi \Rightarrow E_G(\varphi \wedge C_G\varphi)$.

RC1. From $\varphi \Rightarrow E_G(\psi \wedge \varphi)$ infer $\varphi \Rightarrow C_G\psi$.

Historically, in the case of one agent, the system with axioms and rules Prop, K1, MP, and KGen has been called K; adding K2 to K gives us T; adding K4 to T gives us S4; adding K5 to S4 gives us S5; replacing K2 by K3 in S5 gives us KD45. We use the subscript $\mathcal{G}$ to emphasize the fact that we are considering systems with sets of agents coming from $\mathcal{G}$ rather than only one agent and the superscript C to emphasize that we add E1–E3, EGen, C1, and RC1 to the system. In this way, we get the systems $\mathrm{K}^C_{\mathcal{G}}$, $\mathrm{T}^C_{\mathcal{G}}$, and $\mathrm{S4}^C_{\mathcal{G}}$. Thus, $\mathrm{K}^C_{\mathcal{G}}$ consists of Prop, K1, MP, KGen, E1, E2, E3, EGen, C1, and RC1; we get $\mathrm{S4}^C_{\mathcal{G}}$ by adding K2 and K4 to $\mathrm{K}^C_{\mathcal{G}}$. We get $\mathrm{KD45}^C_{\mathcal{G}}$ by adding K3–K5 and E4 to $\mathrm{K}^C_{\mathcal{G}}$ and we get $\mathrm{S5}^C_{\mathcal{G}}$ by adding K2, K4, K5 and E6 to $\mathrm{K}^C_{\mathcal{G}}$.

One of the two main results of this paper shows that each of these axiom systems is sound and complete with respect to an appropriate class of structures. For example, $\mathrm{K}^C_{\mathcal{G}}$ is a sound and complete axiomatization with respect to $\mathcal{M}_{\mathcal{A}}$ and $\mathrm{S5}^C_{\mathcal{G}}$ is a sound and complete axiomatization with respect to $\mathcal{M}^{rst}_{\mathcal{A}}$. In the case that $\mathcal{A}$ is finite, this result is well known (see [FHMV95, HM92]—as mentioned earlier, E is used in the axiomatization instead of E1–E3 and EGen). What is perhaps surprising is that E1–E3 and EGen suffice even if $\mathcal{A}$ is infinite. For example, suppose that $\mathcal{G}$ just consists of the singleton $\mathcal{A}$. In that case, E2 becomes vacuous. Thus, while the axioms force $E_{\mathcal{A}}\varphi$ to imply that each agent in $\mathcal{A}$ knows φ, we have no way of expressing the converse. Indeed, it is easy to construct a structure for the axioms with the standard interpretations of all the K_i relations but a nonstandard one of $E_{\mathcal{A}}$, where all the agents in $\mathcal{A}$ know φ and yet $E_{\mathcal{A}}\varphi$ does not hold. Consider, for example, a structure with a single state s for the language with an infinite set $\mathcal{A}$ of agents. Suppose that every primitive proposition p is true at s, $\mathcal{K}_i$ is empty for all $i \in \mathcal{A}$, and K_i is interpreted in the usual way for all $i \in \mathcal{A}$ (so that $K_i\varphi$ is true at s for all formulas φ). For $E_{\mathcal{A}}$, however, we say that $E_{\mathcal{A}}\varphi$ holds at s if and only if it is provable in, say, $\mathrm{K}_{\mathcal{G}}^{C}$. Of course, there are obviously standard models in which $E_{\mathcal{A}}p$ does not hold and so (by the soundness of the axioms for standard interpretations) $E_{\mathcal{A}}p$ is not provable. Thus, in this interpretation, $E_{\mathcal{A}}p$ does not hold at s while K_ip does for every $i \in \mathcal{A}$. Finally, it is clear that all the axioms of $\mathrm{K}_{\mathcal{G}}^{C}$ are true in this structure. Similar examples can be given to show that E4 and E6 do not follow from the specified other axioms when the set of agents is infinite.

3 The Main Results and a Proof in a Simplified Setting

In this section, we state the two main results of this paper—complete axiomatizations and decision procedures. We then provide a proof of a simpler version of these results that illustrates some of the main ideas. We first state the completeness results.

Theorem 3.1: *For formulas in the language $\mathcal{L}_{\mathcal{G}}^{C}$:*

(a) $\mathrm{K}_{\mathcal{G}}^{C}$ *is a sound and complete axiomatization with respect to* $\mathcal{M}_{\mathcal{A}}$,

(b) $\mathrm{T}_{\mathcal{G}}^{C}$ *is a sound and complete axiomatization with respect to* $\mathcal{M}_{\mathcal{A}}^{r}$,

(c) $\mathrm{S4}_{\mathcal{G}}^{C}$ *is a sound and complete axiomatization with respect to* $\mathcal{M}_{\mathcal{A}}^{rt}$,

(d) $\mathrm{S5}_{\mathcal{G}}^{C}$ *is a sound and complete axiomatization with respect to* $\mathcal{M}_{\mathcal{A}}^{rst}$,

(e) $\mathrm{KD45}_{\mathcal{G}}^{C}$ *is a sound and complete axiomatization with respect to* $\mathcal{M}_{\mathcal{A}}^{elt}$.

Before stating the results regarding complexity, we first show that questions about certain facts regarding sets of the form $G_0 - (G_1 \cup \ldots \cup G_k)$ are reducible to satisfiability. We are not just interested in sets of the form $G_0 - (G_1 \cup \ldots \cup G_k)$ for $G_1, \ldots, G_k \in \mathcal{G}$. For example, when dealing with $\mathcal{M}^{rt}$, it turns out that we are interested in sets H of this form if $|H| = 1$. But if H_1 is such a set, then we are also interested in sets of the form $H_2 = G_0 - (G_1 \cup \ldots G_k \cup H_1)$. And if $|H_2| = 1$, then we can also include H_2 in the union, and so on. The following definition makes this precise.

Definition 3.2: Given a set $\mathcal{J}$ of subsets of $\mathcal{A}$ and an integer $m \geq 1$, define a sequence $\mathcal{J}_0^m, \mathcal{J}_1^m, \ldots$ of sets of subsets of $\mathcal{A}$ inductively as follows. Let $\mathcal{J}_0^m = \mathcal{J}$. Suppose that we have defined $\mathcal{J}_0^m, \ldots, \mathcal{J}_k^m$. Then $\mathcal{J}_{k+1}^m = \mathcal{J} \cup \{G - \cup\mathcal{H} : G \in \mathcal{J},\ \mathcal{H} \subseteq \mathcal{J}_k^m,\ \mathcal{H} \text{ finite}, |G - \cup\mathcal{H}| \leq m\}$. Let $\mathcal{J}^m = \cup_i \mathcal{J}_i^m$; let $\widehat{\mathcal{J}}^m = \{G - \cup\mathcal{H} : G \in \mathcal{J}, \mathcal{H} \subseteq \mathcal{J}^m, \mathcal{H} \text{ finite}\}$. For uniformity, we take $\widehat{\mathcal{J}}^0 = \{G - \cup\mathcal{H} : G \in \mathcal{J}, \mathcal{H} \subseteq \mathcal{J}, \mathcal{H} \text{ finite}\}$. ∎

Let $\mathcal{J}^*$ be the algebra generated by $\mathcal{J}$ (that is, the Boolean combinations of sets in $\mathcal{J}$). It is useful to talk about the length of a description of various sets in $\mathcal{J}^*$ (particularly those in

$\widehat{\mathcal{J}}^m$ for some m). Formally, we assume we have a language whose primitive objects consist of the elements of $\mathcal{J}$ and the symbols $\cup$ and $-$ (for set difference). The length of a description is then the number of symbols of $\mathcal{J}$ that appear in it. Notice that, in general, an element of $\mathcal{J}^*$ may have several different descriptions. We are not always careful to distinguish a set from its description. (We hope that the reader will be able to tell which is intended from context.) We use $l(G)$ to denote the length of the description of $G \in \mathcal{J}^*$.

Let $\mathcal{G}_\mathcal{A} = \mathcal{G} \cup \{\{i\} : i \in \mathcal{A}\}$. Throughout the paper (and, in particular, in the proof of the next proposition), for ease of exposition, we identify $E_{\{i\}}$ with K_i, for $i \in \mathcal{A}$ (which allows us to write E_G for each $G \in \mathcal{G}_\mathcal{A}$).

Proposition 3.3:

(a) The question of whether $|G| > 0$ for $G \in \widehat{\mathcal{G}}^0_\mathcal{A}$ is reducible (in time linear in $l(G)$) to the satisfiability problem for the language $\mathcal{L}^E_\mathcal{G}$ with respect to all of $\mathcal{M}_\mathcal{A}$, $\mathcal{M}^r_\mathcal{A}$, $\mathcal{M}^{rt}_\mathcal{A}$, $\mathcal{M}^{rst}_\mathcal{A}$, and $\mathcal{M}^{elt}_\mathcal{A}$.

(b) The questions of whether $|G| > 0$ and $|G| > 1$ for $G \in \widehat{\mathcal{G}}^1_\mathcal{A}$ are each reducible (in time linear in $l(G)$) to the satisfiability problem for the language $\mathcal{L}^E_\mathcal{G}$ with respect to all of $\mathcal{M}^{rt}_\mathcal{A}$, $\mathcal{M}^{rst}_\mathcal{A}$, and $\mathcal{M}^{elt}_\mathcal{A}$.

(c) For all $m \geq 1$, the question of whether $|G| > m$ for $G \in \widehat{\mathcal{G}}^m_\mathcal{A}$ is reducible (in time linear in $l(G) + m$) to the satisfiability problem for $\mathcal{L}^E_\mathcal{G}$ with respect to $\mathcal{M}^{rst}_\mathcal{A}$ and $\mathcal{M}^{elt}_\mathcal{A}$.

(d) The question of whether $|G_1 \cap \ldots \cap G_k| > 0$, for $G_1, \ldots, G_k \in \mathcal{G}_\mathcal{A}$ is reducible (in time linear in k) to the satisfiability problem for $\mathcal{L}^E_\mathcal{G}$ with respect to $\mathcal{M}^{elt}_\mathcal{A}$.

Proof: For part (a), suppose that $G \in \widehat{\mathcal{G}}^0_\mathcal{A}$. Thus, $G = G_0 - (G_1 \cup \ldots \cup G_k)$ for some $G_0, \ldots, G_k \in \mathcal{G}_\mathcal{A}$. Consider the formula $\varphi_a =_{\rm def} \neg E_{G_0} p \wedge E_{G_1} p \wedge \ldots \wedge E_{G_k} p$, where p is a primitive proposition. Clearly φ_a is satisfiable in $\mathcal{M}_\mathcal{A}$, $\mathcal{M}^r_\mathcal{A}$, $\mathcal{M}^{rt}_\mathcal{A}$, $\mathcal{M}^{rst}_\mathcal{A}$, or $\mathcal{M}^{elt}_\mathcal{A}$ iff $|G_0 - (G_1 \cup \ldots \cup G_k)| > 0$.

For part (b), given G, we construct two formulas $\varphi_{G,p}$ and ψ_G with the following properties.

- $\varphi_{G,p}$ is satisfiable iff $|G| > 0$.
- If $(M, s) \models \varphi_{G,p}$, then $(M, s) \models \neg K_j p$ for some $j \in G$.
- ψ_G is satisfiable iff $|G| > 1$.
- $|\varphi_{G,p}|$ and $|\psi_G|$ are both linear in $l(G)$.

This, of course, suffices to prove the result.

We construct the formulas $\varphi_{G,p}$ by induction on the least h such that $G = G' - \cup\mathcal{H}$ and $\mathcal{H} \subseteq (\mathcal{G}_\mathcal{A})^1_h$. (We are here thinking of G as specified by its description.) If $\mathcal{H} \subseteq (\mathcal{G}_\mathcal{A})^1_0 = \mathcal{G}_A$, suppose that $\mathcal{H} = \{G_1, \ldots, G_k\}$. Then we take $\varphi_{G,p}$ to be $\neg E_{G'} p \wedge E_{G_1} p \wedge \ldots \wedge E_{G_k} p$. This clearly has the desired properties.

Now suppose that $\mathcal{H} \subseteq (\mathcal{G}_\mathcal{A})^1_h$ for $h \geq 1$. Without loss of generality, we can assume that $\mathcal{H} = \{G_1, \ldots, G_{k'}, G_{k'+1}, \ldots, G_k\}$, where $G_1, \ldots, G_{k'} \in \mathcal{G}_\mathcal{A}$ and, for $j = k' + 1, \ldots, k$,

$G_j \in (\mathcal{G}_{\mathcal{A}})^1_h - \mathcal{G}_{\mathcal{A}}$ is of the form $G'_j - \cup\mathcal{H}_j$ with $G'_j \in \mathcal{G}_{\mathcal{A}}$, $\mathcal{H}_j \subseteq (\mathcal{G}_{\mathcal{A}})^1_{h-1}$, and $|G_j| = 1$. Define $\varphi_{G,p}$ as

$$\neg E_{G'}\neg(\neg p \wedge \bigwedge_{j=k'+1}^{k} \varphi_{G_j,p_j}) \wedge E_{G_1}p \wedge \ldots \wedge E_{G_{k'}}p \wedge \bigwedge_{j=k'+1}^{k} E_{G'_j}p_j,$$

where we assume that the sets of primitive propositions that appear in φ_{G_j,p_j}, $j = k'+1,\ldots,k$, are mutually exclusive and do not include p.[2]

Now suppose that $\varphi_{G,p}$ is true at some state s in $M \in \mathcal{M}^{rt}_{\mathcal{A}}$. Then for some $i \in G'$, we must have $(M,s) \models \neg K_i\neg(\neg p \wedge \bigwedge_{j=k'+1}^{k} \varphi_{G_j,p_j})$. We cannot have $i \in G_1 \cup \ldots \cup G_{k'}$, since $(M,s) \models E_{G_j}p$ for $j = 1,\ldots,k'$. Nor can we have $i \in G_j$ for $j = k'+1,\ldots,k$. For suppose that $G_j = \{i_j\}$, $j \in \{k'+1,\ldots,k\}$. Then $(M,s) \models \neg K_i\neg\varphi_{G_j,p_j} \wedge E_{G'_j}p_j$, which implies that $(M,s) \models \neg K_iK_{i_j}p_j \wedge K_{i_j}p_j$. Thus, we cannot have $i = i_j$. It follows that $G \neq \emptyset$.

Conversely, if $G \neq \emptyset$, we show that $\varphi_{G,p}$ is satisfiable in $\mathcal{M}^{rst}_{\mathcal{A}}$ (and hence also in $\mathcal{M}^{rt}_{\mathcal{A}}$ and $\mathcal{M}^{elt}_{\mathcal{A}}$). We actually prove a stronger result. We show that if $G_1,\ldots,G_k$ are nonempty and the formulas $\varphi_{G_1,p_1},\ldots,\varphi_{G_k,p_k}$ involve disjoint sets of primitive propositions, then $\varphi_{G_1,p_1} \wedge \ldots \wedge \varphi_{G_k,p_k}$ is satisfiable in a structure in $\mathcal{M}^{rst}_{\mathcal{A}}$ of a certain form. To make this precise, suppose that $M = (S,\pi,\{\mathcal{K}_i, i \in \mathcal{A}\})$, $s \in S$, S' is a set of states disjoint from S, and $s' \in S'$. We say that M is *embedded in the structure* $M' = (S \cup S', \pi', \{\mathcal{K}'_i, \in \mathcal{A}\})$ *at* (s,s') if

1. $\pi'|_S = \pi$ and $\mathcal{K}'_i|_{S\times S} = \mathcal{K}_i$ for $i \in \mathcal{A}$,
2. if $(t,t') \in \mathcal{K}'_i$ for $t \in S$ and $t' \in S'$, then $t = s$ and $t' = s'$.

We show by induction on h that if $G_1,\ldots,G_k \in (\mathcal{G}_{\mathcal{A}})^1_h$, $|G_i| > 0$ for $i = 1,\ldots,k$, and the formulas $\varphi_{G_1,p_1},\ldots,\varphi_{G_k,p_k}$ involve disjoint sets of primitive propositions, then for all $i_1,\ldots,i_k$ such that $i_j \in G_j$, there exists a structure $M \in \mathcal{M}^{rst}_{\mathcal{A}}$ and a state s in M such that:

1. $(M,s) \models \varphi_{G_1,p_1} \wedge \ldots \wedge \varphi_{G_k,p_k}$,
2. $\exists t_1,\ldots,t_k$ such that $(s,t_j) \in \mathcal{K}_{i_j}$ and $(M,t_j) \models \neg p_j$,
3. $\mathcal{K}_i(s) = \{s\}$ for $i \notin \{i_1,\ldots,i_k\}$,
4. for all structures M' and states s' in M' such that M is embedded in M' at (s,s') and $(M',s') \models p_1 \wedge \ldots \wedge p_k$, we have that $(M',s) \models \varphi_{G_1,p_1} \wedge \ldots \wedge \varphi_{G_k,p_k}$.

If $h = 1$, then it is easy to construct such a structure. Given $i_1,\ldots,i_k$ such that $i_j \in G_j$ (where the i_j are not necessarily distinct) we construct a structure M with states $s,t_1,\ldots,t_k$ (where $t_j = t_{j'}$ if $i_j = i_{j'}$) such that $(M,t_j) \models \bigwedge_{\{j':i_{j'}=i_j\}} \neg p_{j'} \wedge \bigwedge_{\{j':i_{j'}\neq i_j\}} p_{j'}$, $(M,s) \models p_1 \wedge \ldots \wedge p_k$, and $\mathcal{K}_i$ is the least equivalence relation that includes (s,t_j) if $i = i_j$. It is easy to check that M has the required properties. For the inductive step, suppose that we are given $i_1,\ldots,i_k$ such that $i_j \in G_j$. Note that the first conjunct of φ_{G_j,p_j} has the form $\neg E_{G'_j}\neg(\neg p_j \wedge \bigwedge_{k=1}^{m_j} \varphi_{G_{jk},p_{jk}})$. By the induction hypothesis, we can find a structure M_j with state space S_j and a state t_j in S_j with the properties above such that $(M,t_j) \models p_j \wedge \bigwedge_{k=1}^{m_j} \varphi_{G_{jk},p_{jk}}$. If $i_j = i_{j'}$, we can also assume

[2]Here we are implicitly assuming that the set of primitive propositions is infinite, so that this can be done. With more effort, we can prove a similar result even if the set is finite, using the techniques of [Hal95].

without loss of generality that $M_j = M_{j'}$ and $t_j = t_{j'}$. Let S consist of $S_1 \cup \ldots \cup S_k$ together with a new state s. We define $M \in \mathcal{M}_{\mathcal{A}}^{rst}$ so that each of the structures M_j is embedded in M at (s, t_j) and the relation in $\mathcal{K}_{i_j}$ in M is the least equivalence relation that makes this true such that $(s, t_j) \in \mathcal{K}_{i,j}$. We leave it to the reader to check that we can define an interpretation π' with all the required properties. Of course, the fact that $\varphi_{G,p}$ is satisfiable if $|G| > 0$ is now immediate.

Finally, define ψ_G to be $\varphi_{G,p} \wedge E_{G'}(q \wedge (\neg p \Rightarrow \varphi_{G,q}))$, where we assume that the primitive propositions that appear in $\varphi_{G,p}$ and $\varphi_{G,q}$ are disjoint.

We claim that ψ_G is not satisfiable if $|G| \leq 1$. Clearly it is not satisfiable if $|G| = 0$, since $\varphi_{G,p}$ is not. So suppose, by way of contradiction, that $G = \{i\}$ and $(M, s) \models \psi_G$ for some $M \in \mathcal{M}_{\mathcal{A}}^{rt}$. Then, thanks to the properties of $\varphi_{G,p}$ and $\varphi_{G,q}$, we must have $(M, s) \models \neg K_i p \wedge K_i(q \wedge (\neg p \Rightarrow \neg K_i q))$. It is easy to see that this gives us a contradiction. On the other hand, if $|G| > 1$, we can construct a structure satisfying ψ_G as follows. Suppose that $i, j \in G$ and $\varphi_{G,p}$ is of the form

$$\neg E_{G'} \neg(\neg p \wedge \bigwedge_{j=k'+1}^{k} \varphi_{G_j, p_j}) \wedge E_{G_1} p \wedge \ldots \wedge E_{G_{k'}} p \wedge \bigwedge_{j=k'+1}^{k} E_{G'_j} p_j.$$

We know that $|G_{k'}| = \cdots = |G_k| = 1$, so by our previous argument, we can find a structure $M' = (S', \ldots) \in \mathcal{M}_{\mathcal{A}}^{rst}$ and states $s', t' \in S'$ such that $(M, s') \models \varphi_{G,q} \wedge \bigwedge_{j=k'+1}^{k} \varphi_{G_j, p_j}$, $(s', t') \in \mathcal{K}_j$, $(M', t') \models \neg q$, and $\mathcal{K}_i(s') = \{s'\}$. Since p does not appear in $\varphi_{G,q}$, we can assume without loss of generality that $(M, s) \models \neg p$. Now let $M \in \mathcal{M}_{\mathcal{A}}^{rst}$ be a structure whose state space is $S' \cup \{s\}$, where s is a fresh state not in S', such that M' is embedded in M at (s, s'), $(s, s') \in \mathcal{K}_i$, $(M, s) \models p \wedge q$, and $\mathcal{K}_{i'}(s) = \{s\}$ for $i' \neq i$. It is easy to see that $(M, s) \models \psi_G$.

For part (c), we construct formulas $\varphi_{m,G,p}$ such that

- if $(M, s) \models \varphi_{m,G,p}$ for $M \in \mathcal{M}_{\mathcal{A}}^{elt}$ (and hence also for $M \in \mathcal{M}_{\mathcal{A}}^{rst}$), then there exist $m+1$ distinct agents $i_1, \ldots, i_{m+1} \in G$ such that $(M, s) \models \neg K_{i_j} \neg p$, $j = 1, \ldots, m+1$;
- $|\varphi_{m,G,p}| = O(l(G) + m)$;
- if $|G| > m$, then $\varphi_{m,G,p}$ is satisfiable in $\mathcal{M}_{\mathcal{A}}^{rst}$ (and hence in $\mathcal{M}_{\mathcal{A}}^{elt}$).

We first define an auxiliary family of formulas. If $G', G_1, \ldots, G_k \subseteq \mathcal{G}_{\mathcal{A}}$, let $\psi_{m,G',G_1,\ldots,G_k,p}$ be the formula

$$\begin{array}{l} E_{G_1} q_0 \wedge \ldots \wedge E_{G_k} q_0 \wedge \\ \neg E_{G'} \neg(p_0 \wedge p_1 \wedge q_1 \wedge E_{G'}(p_0 \Rightarrow p_1 \wedge q_1)) \wedge \\ \quad \ldots \wedge \neg E_{G'} \neg(p_0 \wedge p_{m+1} \wedge q_{m+1} \wedge E_{G'}(p_0 \Rightarrow p_{m+1} \wedge q_{m+1})) \wedge \\ E_{G'}((p_0 \Rightarrow (p \wedge \neg q_0)) \wedge (q_1 \Leftrightarrow \neg p_2 \wedge q_2) \wedge (q_2 \Leftrightarrow \neg p_3 \wedge q_3) \wedge \ldots \wedge (q_{m+1} \Leftrightarrow \mathit{true})), \end{array}$$

where $p_0, \ldots, p_{m+1}, q_0, \ldots, q_{m+1}$ are fresh primitive propositions distinct from p. Observe that $|\psi_{m,G',G_1,\ldots,G_k,p}|$ is $O(k+m)$. It is easy to check that the last clause forces q_i, for $1 \leq i \leq m$, to be equivalent to $\neg p_{i+1} \wedge \ldots \wedge \neg p_{m+1}$. at least in the worlds G'-reachable in one step. Thus, in these worlds, the formulas $p_i \wedge q_i$, $i = 1, \ldots, m+1$, are mutually exclusive. Clearly if $(M, s) \models \psi_{m,G',G_1,\ldots,G_k,p}$ for $M \in \mathcal{M}_{\mathcal{A}}^{elt}$, then there must be agents $i_1, \ldots, i_{m+1}$ in $G' - (G_1 \cup \ldots \cup G_k)$

such that $(M,s) \models \neg K_{i_j}\neg(p_0 \wedge p_j \wedge q_j \wedge E_{G'}(p_0 \Rightarrow p_j \wedge q_j))$. (Note that we cannot have $i_j \in G' \cap (G_1 \cup \ldots \cup G_k)$ since $(M,s) \models E_{G_j} q_0 \wedge E_{G'}(p_0 \Rightarrow \neg q_0))$. Thus, there must exist states t_j, $j = 1,\ldots,m+1$ such that $(s,t_j) \in \mathcal{K}_{i_j}$ and $(M,t_j) \models p_0 \wedge p_j \wedge q_j \wedge E_{G'}(p_0 \Rightarrow p_j \wedge q_j)$. To see that these agents i_j must be distinct, suppose that $i_j = i_{j'}$ for $j < j'$. By the Euclidean property, we have $(t_j, t_{j'}) \in \mathcal{K}_{i_j}$. Since $(M,t_j) \models E_{G'}(p_0 \Rightarrow p_j \wedge q_j)$, we must have $(M,t_{j'}) \models p_j \wedge q_j$. But since $(M,t_{j'}) \models q_j \Leftrightarrow (\neg p_j \wedge \ldots \wedge \neg p_{m+1})$, this is inconsistent with the fact that $(M,t_{j'}) \models p_{j'}$. Since $(M,s) \models E_{G'}(p_0 \Rightarrow p)$, it follows that $(M,s) \models \neg K_{i_j}\neg p$ for $j = 1,\ldots,m+1$. Conversely, it is easy to see that if $|G' - (G_1 \cup \ldots \cup G_k)| > m$ then $\psi_{m,G',G_1,\ldots,G_k,p}$ is satisfiable in $\mathcal{M}^{rst}_{\mathcal{A}}$. We leave the details to the reader.

We now construct the formulas $\varphi_{m,G,p}$ by induction on the least h such that $G = G' - \cup\mathcal{H}$ and $\mathcal{H} \subseteq (\mathcal{G}_{\mathcal{A}})^m_h$. If $\mathcal{H} = \{G_1,\ldots,G_k\} \subseteq (\mathcal{G}_{\mathcal{A}})^m_0 = \mathcal{G}_A$, then we take $\varphi_{m,G,p} = \psi_{m,G',G_1,\ldots,G_k,p}$. Now suppose that $\mathcal{H} \subseteq (\mathcal{G}_{\mathcal{A}})^m_h$ for $h > 1$. Without loss of generality, we can assume that $\mathcal{H} = \{G_1,\ldots,G_{k'},G_{k'+1},\ldots,G_k\}$, where $G_1,\ldots,G_{k'} \in \mathcal{G}_{\mathcal{A}}$ and, for $j = k'+1,\ldots,k$, $G_j \in (\mathcal{G}_{\mathcal{A}})^m_{h-1}$ is of the form $G'_j - \cup\mathcal{H}_j$ with $G'_j \in \mathcal{G}_{\mathcal{A}}$, $\mathcal{H}_j \subseteq (\mathcal{G}_{\mathcal{A}})^m_{h-1}$, and $|G_j| \le m$. Suppose that $|G_j| = m_j$. By induction, for $j = k'+1,\ldots,k$, we can construct formulas $\varphi_{m_j-1,G_j,p}$ such that if $(M,s) \models \varphi_{m_j-1,G_j,p}$, then for each agent $i \in G_j$, we have $(M,s) \models \neg K_i \neg p$ and the formula $\psi_{m,G',G_1,\ldots,G_{k'},p}$. Without loss of generality, we can assume that, other than p, the sets of primitive propositions mentioned in the formulas $\varphi_{m_j-1,G_j,p}$ are disjoint, and these sets are all disjoint from the set of primitive propositions in $\psi_{m,G',G_1,\ldots,G_{k'},p}$. Let $\varphi_{m,G,p}$ be the formula

$$\psi_{m,G',G_1,\ldots,G_{k'},p'} \wedge \bigwedge_{j=k'+1}^{m} \varphi_{m_j-1,G_j,p} \wedge E_{G'}(p' \Rightarrow E_{G'}\neg p).$$

The argument that this formula has the required properties is almost identical to that for $\psi_{m,G',G_1,\ldots,G_k,p}$; we leave details to the reader.

Finally, for part (d), consider the formula φ_d defined as

$$E_{G_1} p_1 \wedge \ldots \wedge E_{G_{k-1}} p_{k-1} \wedge E_{G_k}(\neg p_1 \vee \ldots \vee \neg p_{k-1}).$$

We leave it to the reader to check that φ_d is satisfiable in $\mathcal{M}^{elt}_{\mathcal{A}}$ iff $G_1 \cap \ldots \cap G_k = \emptyset$. ∎

We already saw that for axiom E2 to be recursive, we need to be able to decide whether $|G_0 - (G_1 \cup \ldots \cup G_k)| \ge 1$ (or, equivalently, whether $G_0 \subseteq G_1 \cup \ldots \cup G_k$) for $G_0,\ldots,G_k \in \mathcal{G}_{\mathcal{A}}$. Proposition 3.3 shows that if there is no recursive algorithm for answering such questions, the satisfiability problem for the logic (even without C_G operators) is also not decidable. For simplicity here, we assume we have oracles that can answer the questions that we need to answer (according to Proposition 3.3) in unit time; we consider the complexity of querying the oracle in more detail in Section 4.5. More precisely, let O_m be an oracle that, for a set $G \in \widehat{\mathcal{G}}^m_{\mathcal{A}}$, tells us whether $|G| > k$, for any $k < m$. (Thus, queries to oracle O_m have the form (G,k).) Let O' be an oracle that tells us whether $G_1 \cap \ldots \cap G_k = \emptyset$, for $G_1,\ldots,G_k \in \mathcal{G}_{\mathcal{A}}$.

Theorem 3.4: *There is a constant $c > 0$ (independent of $\mathcal{A}$) and an algorithm that, given as input a formula $\varphi \in \mathcal{L}^C_{\mathcal{G}}$, decides if φ is satisfiable in $\mathcal{M}_{\mathcal{A}}$ (resp., $\mathcal{M}^r_{\mathcal{A}}$, $\mathcal{M}^{rt}_{\mathcal{A}}$, $\mathcal{M}^{rst}_{\mathcal{A}}$, $\mathcal{M}^{elt}_{\mathcal{A}}$) and runs in time $2^{c|\varphi|}$ given oracle O_0 (resp., O_0, O_1, $O_{|\varphi|}$, both $O_{|\varphi|}$ and O'), where queries to the oracle take unit time. Moreover, if $\mathcal{G}$ contains a subset with at least two elements, then there exists a constant $d > 0$ (independent of $\mathcal{A}$) such that every algorithm for deciding the* *satisfiability of formulas in $\mathcal{M}_\mathcal{A}$ (resp., $\mathcal{M}^r_\mathcal{A}$, $\mathcal{M}^{rt}_\mathcal{A}$, $\mathcal{M}^{rst}_\mathcal{A}$, $\mathcal{M}^{elt}_\mathcal{A}$) runs in time at least $2^{d|\varphi|}$, even given access to oracle O_0 (resp., O_0, O_1, $O_{|\varphi|}$, both $O_{|\varphi|}$ and O'), for infinitely many formulas φ.*

Before proving Theorems 3.1 and 3.4, we prove a somewhat simpler theorem that allows us to both explain intuitively why the results are true and point out some of the difficulties in proving them.

Proposition 3.5: *If there is an oracle that decides if $G = \emptyset$ for each Boolean combination G of elements in $\mathcal{G}_\mathcal{A}$, then, for every formula $\varphi \in \mathcal{L}^C_\mathcal{G}$, we can effectively find a formula φ^σ in a language $\mathcal{L}^C_{\mathcal{G}'}$, where $\mathcal{G}'$ consists of all subsets of a set $\mathcal{A}'$ of at most $2^{|\varphi|}$ agents, such that $|\varphi^\sigma| = |\varphi|$ and φ is satisfiable in $\mathcal{M}_\mathcal{A}$ iff φ^σ is satisfiable in $\mathcal{M}_{\mathcal{A}'}$.*

Proof: Given φ, let $\mathcal{G}_\varphi$ be the set of subsets G of agents such that E_G or C_G appears in φ. (Recall that we are identifying K_i with $E_{\{i\}}$, so that $\{i\} \in \mathcal{G}_\varphi$ if K_i appears in φ.) Note that $|\mathcal{G}_\varphi| \leq |\varphi|$.

Suppose that $\mathcal{G} = \{G_1, \ldots, G_N\}$. An *atom over* $\mathcal{G}$ is a nonempty set of the form $G'_1 \cap \ldots \cap G'_N$, where $G'_i = G_i$ or $G'_i = \overline{G_i}$. Clearly there are at most 2^N atoms over $\mathcal{G}$. Let $\mathcal{A}'$ consist of the nonempty atoms over $\mathcal{G}_\varphi$. Note that $|\mathcal{A}'| \leq 2^{|\varphi|}$. Define $\sigma : \mathcal{A} \to \mathcal{A}'$ by taking $\sigma(i)$ to be the unique atom over $\mathcal{G}_\varphi$ containing i. We extend σ to a map from $2^\mathcal{A} \to 2^{\mathcal{A}'}$ by taking $\sigma(G) = \{\sigma(i) : i \in G\}$ $(= \{H \in \mathcal{G}_\varphi : H \subseteq G\})$. Translate φ to φ^σ by replacing all occurrences of E_G and C_G in φ by $E_{\sigma(G)}$, and $C_{\sigma(G)}$, respectively. Clearly $|\varphi| = |\varphi^\sigma|$. (Note that it is important here that we take the length of E_G and C_G to be independent of G.)

If φ is satisfiable, let (M, s) witness that fact. Convert M into a structure M^σ over $\mathcal{A}'$ with the same state space by setting $(s,t) \in \mathcal{K}_A$ iff $(s,t) \in \cup_{j \in A} \mathcal{K}_j$ for each $A \in \mathcal{A}'$. An easy induction shows that for every formula ψ with sets (of agents) chosen from $\mathcal{G}_\varphi$, we have $(M,s) \models \psi$ if and only if $(M^\sigma, s) \models \psi^\sigma$. The only point that needs any comment is that E_G (and so also C_G) has the same meaning in M (in terms of reachability) as $E_{\sigma(G)}$ ($C_{\sigma(G)}$) in M^σ, by the definition of $\sigma(G)$ and the $\mathcal{K}_A$ relations. Thus $(M^\sigma, s) \models \varphi^\sigma$ as required.

For the other direction, suppose that $(M', s) \models \varphi^\sigma$ for some structure M' over $\mathcal{A}'$. We define a structure M over $\mathcal{A}$ by defining $\mathcal{K}_i = \mathcal{K}_{\sigma(i)}$. Again an easy induction shows that for every formula ψ with sets chosen from $\mathcal{G}_\varphi$, $(M', s) \models \psi$ if and only if $(M, s) \models \psi^\sigma$. Once again, the only point to notice is that E_G (and so also C_G) has the same meaning in M' (in terms of reachability) as $E_{\sigma(G)}$ ($C_{\sigma(G)}$) in M by the definition of $\sigma(G)$ and the relations $\mathcal{K}_j$. Thus $(M, s) \models \varphi$ as required. ∎

Corollary 3.6: *Given an oracle that decides, for each Boolean combination G of elements in $\mathcal{G}_\mathcal{A}$, whether $G = \emptyset$, there is a constant $c > 0$ (independent of $\mathcal{A}$) and an algorithm that, given as input a formula $\varphi \in \mathcal{L}^C_\mathcal{G}$, decides if $\varphi \in \mathcal{L}^C_\mathcal{G}$ is satisfiable in $\mathcal{M}_\mathcal{A}$ and runs in time $2^{c2^{|\varphi|}}$.*

Proof: Clearly, to check if φ is satisfiable, it suffices to check if φ^σ is satisfiable. In [HM92], there is an exponential time algorithm for checking satisfiability. However, this algorithm presumes that the set of agents is fixed. A close look at the algorithm actually shows that it runs in time $2^{cm|\varphi|}$, where m is the number of agents. In our translation, the set of agents is exponential in $|\varphi|$, giving us a double-exponential time algorithm. ∎

Corollary 3.7: *If $\mathcal{G}$ is closed under intersection and complementation, then $\mathrm{K}_{\mathcal{G}}^{C}$ is a sound and complete axiomatization for the language $\mathcal{L}_{\mathcal{G}}^{C}$ with respect to $\mathcal{M}_{\mathcal{A}}$.*

Proof: Soundness is straightforward, so we focus on completeness. Suppose that φ is valid. By Proposition 3.5, so is φ^{σ}. Since $\mathcal{A}'$ is finite, $\mathrm{K}_{\mathcal{G}'}^{C}$ is a complete axiomatization for $\mathcal{L}_{\mathcal{G}'}^{C}$ with respect to $\mathcal{M}_{\mathcal{A}'}$. Thus, $\mathrm{K}_{\mathcal{G}'}^{C} \vdash \varphi^{\sigma}$. We can translate this proof step by step to a proof of φ in $\mathrm{K}_{\mathcal{G}}^{C}$. We simply replace every formula ψ that appears in the proof of φ^{σ} by ψ^{τ}, where ψ^{τ} is obtained by replacing each occurrence of K_A in ψ by E_A unless $A = \{i\}$ is a singleton, in which case we replace K_A by K_i, and replacing each occurrence of E_G, and C_G in ψ by $E_{\cup G}$, and $C_{\cup G}$, respectively. Since we have assumed $\mathcal{G}$ is closed under complementation and intersection, it is also closed under union, and hence ψ^{τ} is a formula in $\mathcal{L}_{\mathcal{G}}^{C}$.

It is easy to check that the translated proof is still a proof over the language $\mathcal{L}_{\mathcal{G}}^{C}$: Tautologies become tautologies as $(\varphi \vee \psi)^{\tau} = \varphi^{\tau} \vee \psi^{\tau}$ and similarly for negations. Instances of MP in the proof of φ^{σ} become instances of MP in the proof of φ because $(\varphi \rightarrow \psi)^{\tau} = \varphi^{\tau} \rightarrow \psi^{\tau}$. Instances of KGen in the proof of φ^{σ} become instances of EGen or KGen in the proof of φ; similarly, instances of K1 are converted to instances of K1 or E1. It is easy to see that instances of E1, E2, E3, EGen, C1, and RC1 are converted to legitimate instances of the same axiom. ∎

While Corollaries 3.6 and 3.7 are close to our desired theorems, they also make clear the difficulties we need to overcome in order to prove Theorems 3.1 and 3.4. Specifically,

- we need to cut the complexity down from double-exponential to single exponential;
- we need to prove completeness without assuming that $\mathcal{G}$ is closed under complementation and intersection;
- we want to use an oracle that tests only whether a set of the form $G_0 - (G_1 \cup \ldots \cup G_k)$ is nonempty, rather than one that applies to arbitrary Boolean combinations;
- we want to extend these results to the case that the $\mathcal{K}_i$ relations satisfy properties like transitivity.

With regard to the last point, while in general it is relatively straightforward to extend completeness and complexity results to deal with relations that have properties like transitivity, it is not so straightforward in this case. For example, even if $M \in \mathcal{M}_{\mathcal{A}}^{rt}$, the relations in the structure M^{σ} constructed in Proposition 3.5 are not necessarily transitive. As shown in Proposition 3.3, we need a different oracle to deal with transitivity.

4 Proving the Main Results

In this section, we prove Theorems 3.1 and 3.4. The structure of the proof is similar to that of Corollaries 3.6 and 3.7; we describe step by step the modifications required to deal with the problems raised in the previous section. It is convenient to split the proof into four cases, depending on the class of structures considered.

4.1 The Proof for $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}^r_{\mathcal{A}}$

In Proposition 3.5 we showed that we could translate a formula φ to a formula φ^σ such that φ was satisfiable in $\mathcal{M}_{\mathcal{A}}$ iff φ^σ was satisfiable in $\mathcal{M}_{\mathcal{A}'}$, where $\mathcal{A}'$ consisted of the atoms over $\mathcal{G}_\varphi$. Our goal is to maintain the translation idea, but use as our target set of agents a set whose elements we can determine with the oracles at our disposal (for testing the nonemptiness of certain set differences). As a first step, we try to abstract the key ingredients of Proposition 3.5. Suppose that we have a set $\mathcal{A}'$ of agents and a partial map $\sigma : \mathcal{A} \to \mathcal{A}'$. Again, we can extend σ to a map from $2^{\mathcal{A}}$ to $2^{\mathcal{A}'}$: $\sigma(G) = \{\sigma(i) : i \in \mathcal{G}\}$. Given a formula φ, let φ^σ be the formula that results by replacing all the occurrences of G in φ by $\sigma(G)$. In Proposition 3.5, $\mathcal{A}'$ is the set of atoms over $\mathcal{G}_\varphi$ and $\sigma(i)$ is the unique atom containing i. We were able to show that, for that choice of $\mathcal{A}'$ and σ, the formulas φ and φ^σ were equisatisfiable. What does it take to obtain such a result in general? The following result shows that we need to be able to find a mapping $\tau : \mathcal{A}' \to 2^{\mathcal{A}} - \{\emptyset\}$ with one key property.

Proposition 4.1: *Given a formula φ and a partial map $\sigma : \mathcal{A} \to \mathcal{A}'$ such that $\sigma(G) \neq \emptyset$ for all $G \in \mathcal{G}_\varphi$, suppose that there is a mapping $\tau : \mathcal{A}' \to 2^{\mathcal{A}} - \{\emptyset\}$ such that for all $G \in \mathcal{G}_\varphi$, we have $\cup\{\tau(A) : A \in \sigma(G)\} = G$. Then φ is satisfiable in $\mathcal{M}_{\mathcal{A}}$ (resp., $\mathcal{M}^r_{\mathcal{A}}$) iff φ^σ is satisfiable in $\mathcal{M}_{\mathcal{A}'}$ (resp., $\mathcal{M}^r_{\mathcal{A}'}$).*

Proof: Given φ and σ, suppose there exists a mapping τ with the property above. We show that φ and φ^σ are equisatisfiable.

First suppose that $(M, s) \models \varphi$, where $M \in \mathcal{M}_{\mathcal{A}}$. We convert $M = (S, \pi, \{\mathcal{K}_i : i \in \mathcal{A}\})$ into a structure $M' = (S, \pi, \{\mathcal{K}_A : A \in \mathcal{A}'\})$ by defining $\mathcal{K}_A = \cup\{\mathcal{K}_i : i \in \tau(A)\}$. Notice that the assumed property of τ implies that for all $G \in \mathcal{G}_\varphi$, we have

$$\cup_{A\in\sigma(G)}\mathcal{K}_A = \cup_{A\in\sigma(G)} \cup_{i\in\tau(A)} \mathcal{K}_i = \cup_{i\in G}\mathcal{K}_i.$$

An easy induction on the structure of ψ now shows that $(M, t) \models \psi$ if and only if $(M', t) \models \psi^\sigma$ for all $t \in S$ and all formulas $\psi \in \mathcal{L}^C_{\mathcal{G}_\varphi}$. Also note that if $M \in \mathcal{M}^r_{\mathcal{A}}$, then $M' \in \mathcal{M}^r_{\mathcal{A}'}$ (since the union of reflexive relations is reflexive).

For the opposite direction, suppose $(M', s) \models \varphi^\sigma$ for some $M' = (S, \pi, \{\mathcal{K}_A : A \in \mathcal{A}'\}) \in \mathcal{M}_{\mathcal{A}'}$. Define $M = (S, \pi, \{\mathcal{K}_i : i \in \mathcal{A}\}) \in \mathcal{M}_{\mathcal{A}}$ by setting $\mathcal{K}_i = \mathcal{K}_{\sigma(i)}$ if $\sigma(i)$ is defined and the empty relation otherwise. Note that for all $G \in \mathcal{G}_\varphi$ we have

$$\cup_{i\in G}\mathcal{K}_i = \cup_{i\in G}\mathcal{K}_{\sigma(i)} = \cup_{A\in\sigma(G)}\mathcal{K}_A.$$

Again, an easy induction on the structure of ψ shows that $(M, t) \models \psi$ if and only if $(M', t) \models \psi^\sigma$ for all $t \in S$ and all formulas $\psi \in \mathcal{L}^C_{\mathcal{G}_\varphi}$.

If $M' \in \mathcal{M}^r_{\mathcal{A}}$, we modify the construction slightly by taking $\mathcal{K}_i = \{(t, t) : t \in S\}$ if $\sigma(i)$ is undefined. Since $\sigma(G) \neq \emptyset$ for $G \in \mathcal{G}_\varphi$, it is easy to check that we still have $\cup_{i\in G}\mathcal{K}_i = \cup_{i\in G}\mathcal{K}_{\sigma(i)}$, so the modified construction works for the reflexive case. ∎

For the mapping σ of Proposition 3.5 we can take τ to be the identity, but this requires an oracle for nonemptiness of atoms. We now show how to choose $\mathcal{A}'$ and define maps σ and τ in a way that requires only information about whether sets of the form $G_0 - (G_1 \cup \ldots \cup G_k)$ are empty.

Definition 4.2: Given a set $\mathcal{G}$ of sets of agents and $G \in \mathcal{G}$, a set $\mathcal{H} \subseteq \mathcal{G}$ is a G-*maximal* subset of $\mathcal{G}$ if $G - \cup\mathcal{H} \neq \emptyset$ and $G - ((\cup\mathcal{H}) \cup G') = \emptyset$ for all $G' \in \mathcal{G} - \mathcal{H}$. Let $\mathcal{R}(\mathcal{G}) = \{(G, \mathcal{H}) : G \in \mathcal{G}, \mathcal{H}$ is a G-maximal subset of $\mathcal{G}\}$. ∎

Note that we can check whether $\mathcal{H}$ is a G-maximal subset of $\mathcal{G}$ by doing at most $|\mathcal{G}|$ tests of the form $(G - \cup\mathcal{H}') = \emptyset$, and we can find all pairs $(G, \mathcal{H})$ in $\mathcal{R}(\mathcal{G}_\varphi)$ by doing at most $|\mathcal{G}|2^{|\mathcal{G}|-1}$ such tests.

The following lemma gives some technical properties of $\mathcal{R}(\mathcal{G})$ that will be used frequently.

Lemma 4.3: *Suppose that $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{G})$ for some set $\mathcal{G}$ of subsets of $\mathcal{A}$.*

(a) $G - \cup\mathcal{H}$ is an atom over $\mathcal{G}$ and, in fact, $G - \cup\mathcal{H} = \cap(\mathcal{G} - \mathcal{H}) \cap (\cap_{H \in \mathcal{H}} \overline{H})$.

(b) If $(G', \mathcal{H}) \in \mathcal{R}(\mathcal{G})$, then $(G - \bigcup\mathcal{H}) = (G' - \bigcup\mathcal{H})$.

(c) If $(G', \mathcal{H}') \in \mathcal{R}(\mathcal{G})$ and $\mathcal{H} \neq \mathcal{H}'$, then $(G - \bigcup\mathcal{H}) \cap (G' - \bigcup\mathcal{H}') = \emptyset$.

Proof: For part (a), first observe that since $\mathcal{H}$ is a G-maximal subset of $\mathcal{G}$, for $H \notin \mathcal{H}$, we have $G - \cup(\mathcal{H} \cup \{H\}) = \emptyset$; i.e., $G - \cup\mathcal{H} \subseteq H$. Thus, if $H \notin \mathcal{H}$, we have $G - \cup\mathcal{H} = (G \cap H) - \cup\mathcal{H}$. Thus, $G - \cup\mathcal{H} = G \cap (\cap_{H \in \mathcal{H}} \overline{H}) = \cap(\mathcal{G} - \mathcal{H}) \cap (\cap_{H \in \mathcal{H}} \overline{H})$, as desired. By definition, $G - \cup\mathcal{H}$ is an atom over $\mathcal{G}$.

Part (b) is immediate from part (a), since it is clear that $G - \cup\mathcal{H}$ is independent of G and depends only on $\mathcal{H}$.

For part (c), suppose that $\mathcal{H} \neq \mathcal{H}'$. Without loss generality, there is some $H \in \mathcal{H} - \mathcal{H}'$. It follows immediately from part (a) that $G - \cup\mathcal{H}$ and $G' - \cup\mathcal{H}'$ are distinct atoms (hence disjoint), since $G - \cup\mathcal{H} \subseteq \overline{H}$ and $G' \cup \mathcal{H}' \subseteq H$. ∎

If $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{G})$, let $A^{\mathcal{G}}_{\mathcal{H}}$ denote the atom associated with $\mathcal{H}$ defined in Lemma 4.3(a). It is independent of G by Lemma 4.3(b). We omit $\mathcal{G}$, writing simply $A_{\mathcal{H}}$, when it is clear from the context which set $\mathcal{G}$ we have in mind.

We now show how to define a translation satisfying the hypotheses of Proposition 4.1 using the elements of $\mathcal{R}(\mathcal{G}_\varphi)$ identified according to the second coordinate alone.

Given a formula φ, let $\mathcal{A}^\varphi = \{\mathcal{H} : \exists G[(G, \mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi)]\}$. Define $\sigma_1 : \mathcal{A} \to \mathcal{A}^\varphi$ by setting $\sigma_1(i) = \mathcal{H}$ if $i \in A_{\mathcal{H}}$ (as defined after Lemma 4.3) and undefined otherwise. As before, we extend σ_1 to $2^{\mathcal{A}}$ by defining $\sigma_1(G) = \{\sigma_1(i) : i \in G\}$.

Lemma 4.4: *Define $\tau : \mathcal{A}^\varphi \to 2^{\mathcal{A}}$ by setting $\tau(\mathcal{H}) = \cap(\mathcal{G}_\varphi - \mathcal{H})$. Then*

(a) $\sigma_1(G) = \{\mathcal{H} \in \mathcal{A}^\varphi : \exists G' \in \mathcal{G}_\varphi((G', \mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi)), G \notin \mathcal{H}\}$,

(b) $\sigma_1(G) \neq \emptyset$ for $G \in \mathcal{G}_\varphi$,

(c) $\tau(\mathcal{H}) \neq \emptyset$ for $\mathcal{H} \in \mathcal{A}^\varphi$,

(d) $\cup\{\tau(\mathcal{H}) : \mathcal{H} \in \sigma_1(G)\} = G$.

Proof: For part (a), first suppose that $G \notin \mathcal{H}$ and $(G', \mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi)$ for some $G' \in \mathcal{G}_\varphi$. Then by Lemma 4.3(a), it follows that $A_\mathcal{H} \subseteq G$. Since $A_\mathcal{H} \neq \emptyset$, there is some $i \in A_\mathcal{H}$. Since $i \in G$ and $\sigma_1(i) = \mathcal{H}$, it follows that $\mathcal{H} \in \sigma_1(G)$. For the opposite inclusion, suppose that $\mathcal{H} \in \sigma_1(G)$. Then $\mathcal{H} = \sigma_1(i)$ for some $i \in G \cap A_\mathcal{H}$. Since $G \cap A_\mathcal{H} \neq \emptyset$, it follows from the definition of $A_\mathcal{H}$ that $G \notin \mathcal{H}$.

For part (b), given G, note that there must be some G-maximal subset $\mathcal{H}$. Thus, $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi)$. Since $G - \cup\mathcal{H} \neq \emptyset$, we must have $G \notin \mathcal{H}$. By part (a), $\mathcal{H} \in \sigma_1(G)$, so $\sigma_1(G) \neq \emptyset$.

For part (c), suppose that $\mathcal{H} \in \mathcal{A}^\varphi$. Then there exists some G such that $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi)$, and hence $G - \cup\mathcal{H} \neq \emptyset$. It suffices to show that $\cap(\mathcal{G}_\varphi - \mathcal{H}) \supseteq G - \cup\mathcal{H}$. Since $G - ((\cup\mathcal{H}) \cup G') = \emptyset$ for all $G' \in \mathcal{G}_\varphi - \mathcal{H}$, it follows that $G - \cup\mathcal{H} \subseteq G'$ for each $G' \in \mathcal{G}_\varphi - \mathcal{H}$. Thus, $\cap(\mathcal{G}_\varphi - \mathcal{H}) \supseteq G - \cup\mathcal{H}$.

For part (d), we first show that $\cup\{\tau(\mathcal{H}) : \mathcal{H} \in \sigma_1(G)\} \subseteq G$. Note that if $\mathcal{H} \in \sigma_1(G)$, then by part (a), $G \in \mathcal{G}_\varphi - \mathcal{H}$. Thus, $\tau(\mathcal{H}) = \cap(\mathcal{G}_\varphi - \mathcal{H}) \subseteq G$. For the oppposite containment, suppose that $i \in G$. Let $\mathcal{H}^i = \{G' \in \mathcal{G}_\varphi : i \notin G'\}$. Since $i \in G - \cup\mathcal{H}^i$, there must be a G-maximal subset $\mathcal{H}$ of $\mathcal{G}_\varphi$ containing $\mathcal{H}^i$. By part (a), we have $\mathcal{H} \in \sigma_1(G)$. Moreover, since $\mathcal{H}^i \subseteq \mathcal{H}$, for all $H' \in \mathcal{G}_\varphi - \mathcal{H}$, we have $i \in H'$. Thus, $i \in \cap(\mathcal{G}_\varphi - \mathcal{H})$. It follows that $i \in \cup_{\mathcal{H} \in \sigma_1(G)} \cap (\mathcal{G}_\varphi - \mathcal{H})$, as desired. ∎

Since $|\mathcal{A}| \leq 2^{|\varphi|}$, we have now reduced satisfiability with infinitely many agents to satisfiability with finitely many agents, at least for $\mathcal{M}_\mathcal{A}$ and $\mathcal{M}^r_\mathcal{A}$, using only tests that we know we need to be able to perform in any case. We next must deal with the problem we observed in the proof of Corollary 3.6, that is, there may be exponentially many agents in the subgroups mentioned in φ^{σ_1}. This is done in the following result. In this result, we assume that the complexity of checking whether $i \in G$ is no worse than linear in $|\mathcal{A}|$. While we do not assume this in general, it is true for the $\mathcal{A}'$ and sets G that arise in the translation of Proposition 4.1, which suffices for our application of the result to the proof of Theorem 3.4.

Theorem 4.5: *If $\mathcal{A}$ is finite and there is an algorithm for deciding if $i \in G$ for $G \in \mathcal{G}$ that runs in time linear in $|\mathcal{A}|$, then there is a constant $c > 0$ (independent of $\mathcal{A}$) and an algorithm that, given as input a formula $\varphi \in \mathcal{L}^C_\mathcal{G}$, decides if φ is satisfiable in $\mathcal{M}_\mathcal{A}$ (resp., $\mathcal{M}^r_\mathcal{A}$) and runs in time $O(|\mathcal{A}|2^{c|\varphi|})$.*

Proof: We first present an algorithm that decides if φ is satisfiable in $\mathcal{M}_\mathcal{A}$; we then show how to modify it to deal with $\mathcal{M}^r_\mathcal{A}$. The algorithm is just a slight modification of standard decision procedures [FHMV95, HM92]. (Far more serious modifications are needed to prove the analogous result for the $\mathcal{M}^{rt}_\mathcal{A}$, $\mathcal{M}^{rst}_\mathcal{A}$, and $\mathcal{M}^{elt}_\mathcal{A}$; see Theorems 4.9, 4.16, and 4.20.)

Let $Sub(\varphi)$ be the set of subformulas of φ together with $E_G(\psi \wedge C_G\psi)$ and $\psi \wedge C_G\psi$ for each subformula $C_G\psi$ of φ. $Sub^+(\varphi)$ consists of the formulas in $Sub(\varphi)$ and their negations. An easy induction on $|\varphi|$ shows that $|Sub(\varphi)| \leq |\varphi|$, so $|Sub^+(\varphi)| \leq 2|\varphi|$. (Here we need to use the fact that we take the length of C_G to be 3.)

Let S^1 consist of all subsets s of $Sub^+(\varphi)$ that are *maximally consistent* in that (a) for each formula $\psi \in Sub(\varphi)$, either $\psi \in s$ or $\neg\psi \in s$, (b) they are propositionally consistent (for example, we cannot have all of $\psi \wedge \psi'$, $\neg\psi$, and $\neg\psi'$ in s), and (c) they contain $E_G(\psi \wedge C_G\psi)$ iff they contain $C_G\psi$. Note that there are at most $2^{|\varphi|}$ sets in $S^1(\varphi)$.

For $s \in S^1$ and $G \in \mathcal{G}_\mathcal{A}$, we define $s/E_G = \{\psi : E_G\psi \in s\}$ (again, we identify K_i with $E_{\{i\}}$). Define $s/\overline{K_i} = \cup_{i \in G}(s/E_G)$. Define a binary relation $\mathcal{K}_i$ on S^1 for each $i \in \mathcal{A}$ by taking

$(s,t) \in \mathcal{K}_i$ iff $s/\overline{K_i} \subseteq t$. We now define a sequence S^j of subsets of S^1. Suppose that we have defined $S^1, \ldots, S^j$. S^{j+1} consists of all states in S^j that *seem consistent*, in that the following two conditions hold:

1. If $\neg E_G\psi \in s$, then there is some $t \in S^j$ such that $(s,t) \in \cup_{i \in G}\mathcal{K}_i$ and $\neg\psi \in t$.
2. If $\neg C_G\psi \in s$, then there is some $t \in S^j$ such that t is G-reachable from s in S^j and $\neg\psi \in t$.

If $S^j \neq S^{j+1}$ then we continue the construction. Otherwise the construction terminates; in this case, the algorithm returns "φ is satisfiable" if $\varphi \in s$ for some state $s \in S^{j+1}$ and returns "φ is unsatisfiable" otherwise.

Since $S^j \supseteq S^{j+1}$, S^1 has at most $2^{|\varphi|}$ elements, and there are $|\mathcal{A}|$ relations, it is easy to see that the whole procedure can be carried out in time $O(|\mathcal{A}|2^{c|\varphi|})$ for some $c > 0$.

It remains to show that the algorithm is correct. First suppose that φ is satisfiable. In that case, $(M, s_0) \models \varphi$ for some structure $M = (S, \pi, \{\mathcal{K}'_i : i \in \mathcal{A}\}) \in \mathcal{M}_\mathcal{A}$. We can associate with each state $s \in S$ the state s^* in S^1 consisting of all the formulas $\psi \in \mathit{Sub}(\varphi)$ such that $(M, s) \models \psi$. It is easy to see that if $(s,t) \in \mathcal{K}'_i$ then $(s^*, t^*) \in \mathcal{K}_i$. A straightforward induction shows that the states s^* for $s \in S$ always seem consistent, and thus are in S^j for all j. Moreover, $\varphi \in s_0^*$. Thus, the algorithm declares that φ is satisfiable, as desired.

Conversely, suppose that the algorithm declares that φ is satisfiable. We construct a structure $M = (S, \pi, \{\mathcal{K}'_i : i \in \mathcal{A}\})$ over $\mathcal{A}$ and Φ in which φ is satisfied as follows. Let j be the stage at which the algorithm terminates. Let $S = S^j$. Define π so that $\pi(s)(p) = \mathbf{true}$ iff $p \in s$, for $s \in S$ and $p \in \Phi$. For each $i \in \mathcal{A}$, we take $\mathcal{K}'_i$ to be the restriction of $\mathcal{K}_i$ to S^j. A straightforward induction on the structure of formulas shows that for all formulas $\psi \in \mathit{Sub}(\varphi)$ and states $s \in S$, we have $(M, s) \models \psi$ iff $\psi \in s$. (The cases for $E_G\psi$ and $C_G\psi$ use the appropriate clauses of the definition of seeming inconsistent and the choice of j.) Since $\varphi \in s$ for some $s^* \in S$, it follows that $(M, s^*) \models \varphi$, so φ is satisfiable.

To deal with $\mathcal{M}^r_\mathcal{A}$, the only change necessary is that in going from S^1 to S^2 in the construction, we also eliminate $s \in S^1$ if $(s,s) \notin \mathcal{K}_i$ for some $i \in \mathcal{A}$. This guarantees that the $\mathcal{K}_i$ relations are reflexive. The remainder of the proof goes through unchanged. ∎

Proof of Theorem 3.4 for $\mathcal{M}_\mathcal{A}$ and $\mathcal{M}^r_\mathcal{A}$: The deterministic exponential time lower bound in Theorem 3.4 follows from the lower bound in the case where $\mathcal{A}$ is finite, which is proved in [HM92, Theorem 6.19] using techniques developed by Fischer and Ladner [FL79] for PDL. The sets G that arise in the lower bound proof have cardinality 2, so oracles are of no help here.

For the upper bound, suppose that we are given a formula φ. We first compute the set $\mathcal{R}(\mathcal{G}_\varphi)$. This can be done with at most $|\varphi|2^{|\varphi|}$ calls to oracle O_0, since $|\mathcal{G}_\varphi| \leq |\varphi|$ and we need only check, for each $G \in \mathcal{G}_\varphi$ and $\mathcal{H} \subseteq \mathcal{G}_\varphi$, whether $G - \mathcal{H} = \emptyset$.

Consider the mapping σ_1 of Lemma 4.4. By part (a) of Lemma 4.4, we can compute the formula φ^{σ_1} using $\leq |\varphi|2^{|\varphi|}$ calls to oracle O_0. By Proposition 4.1 and Lemma 4.4, the formulas φ and φ^{σ_1} are equisatisfiable. By Theorem 4.5, we can decide if φ^{σ_1} is satisfiable in time $O(2^{c|\varphi|})$ for some $c > 0$ (since $|\varphi^{\sigma_1}| = |\varphi|$ and the set $\mathcal{A}$ of agents that appear in φ^{σ_1} has size at most $2^{|\varphi|}$). ∎

We now want to prove Theorem 3.1 in the case of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}^r_{\mathcal{A}}$. The idea is the same as that of Corollary 3.7. If φ is valid, then so is φ^{σ_1}. We can then appeal to completeness in the case of finitely many agents to get a proof of φ^{σ_1} that we can then "pull back" to a proof of φ. There is only one difficulty that we encounter when trying to put this idea into practice. Exactly how do we pull back the proof? For example, suppose that the proof of φ^{σ_1} involves a formula ψ with an operator $K_{\mathcal{H}}$. In general, there will be many agents $i \in \mathcal{A}$ such that $\sigma_1(i) = \mathcal{H}$. One option is to replace $K_{\mathcal{H}}$ by $E_{\sigma_1^{-1}(\mathcal{H})}$, that is, replace $\mathcal{H}$ by all i such that $\sigma_1(i) = \mathcal{H}$. (This is what was done in the proof of Corollary 3.7.) The problem with this is that there is no guarantee that the resulting set is in $\mathcal{G}$. Alternatively, we could replace $K_{\mathcal{H}}$ by K_i for some i such that $\sigma_1(i) = \mathcal{H}$. But if so, which one?

We actually take the latter course here. We solve the problem of which i to choose by showing that there is a proof of φ^{σ_1} in which the only modal operators that arise in any formula used in the proof are modal operators that appear in φ^{σ_1} (Lemma 4.7). For these operators, there is a canonical way to do the replacement (Lemma 4.6). While it may seem almost trivial that the only operators that should be needed in the proof of φ^{σ_1} are ones that already appear in the formula, this is not the case for the standard completeness proof [FHMV95, HM92], since in the proof of the validity of a formula of the form $E_G\psi$, the modal operators K_i are used for $i \in G$, although these operators may not appear in ψ. It is important that we use the axioms E1 and E2 in doing the proof, rather than the axiom E; otherwise the result would not hold. Indeed, the result does not quite hold in the case of $\mathrm{T}^C_{\mathcal{G}}$; we need to augment it with E5.

Lemma 4.6: *The mapping σ_1 (when viewed as a map with domain $2^{\mathcal{A}}$) is injective on $\mathcal{G}_\varphi$.*

Proof: Suppose that $G \neq G'$. Without loss of generality, suppose that $i \in G - G'$. Then there is a G-maximal set $\mathcal{H}$ that includes G'. By Lemma 4.4(a), we have $\mathcal{H} \in \sigma_1(G)$. Since $G' \in \mathcal{H}$, it follows from Lemma 4.4(a) that $\mathcal{H} \notin \sigma_1(G')$. Thus, $\sigma_1(G) \neq \sigma_1(G')$. ∎

For the next lemma, we write $AX \vdash_\varphi \psi$ if there is a proof of φ in AX that involves only modal operators that appear in φ. Let $(\mathrm{T}^C_{\mathcal{G}})^+$ consist of $\mathrm{T}^C_{\mathcal{G}}$ augmented with the axiom E5. Although E5 follows from E1 and K2, using E5 allows us to be able to write proofs of φ that use only the modal operators in φ.

Lemma 4.7: *If $\mathcal{A}$ is finite and $\varphi \in \mathcal{L}^C_{\mathcal{G}}$ is valid with respect to $\mathcal{M}_{\mathcal{A}}$ (resp., $\mathcal{M}^r_{\mathcal{A}}$), then $\mathrm{K}^C_{\mathcal{G}} \vdash_\varphi \varphi$ (resp., $(\mathrm{T}^C_{\mathcal{G}})^+ \vdash_\varphi \varphi$).*

Proof: We first consider the case of $\mathcal{M}_{\mathcal{A}}$. Since φ is valid, $\neg\varphi$ is not satisfiable. That means, when we apply the construction in the proof of Theorem 4.5 to $\neg\varphi$, all the sets containing $\neg\varphi$ are eliminated. For each state $s \in S^1$, let φ_s be the conjunction of all the formulas in s.

We prove the result by showing, by induction on j, that

$$\text{if a state } s \in S^j \text{ does not seem consistent, then } \varphi_s \text{ is } \mathrm{K}^C_{\mathcal{G}}\text{-inconsistent, i.e., } \mathrm{K}^C_{\mathcal{G}} \vdash_\varphi \neg\varphi_s. \quad (1)$$

To see that (1) suffices to prove the lemma, note that standard propositional reasoning (i.e., using Prop and MP) shows that, for any formula $\psi \in Sub(\neg\varphi)$,

$$\mathrm{K}^C_{\mathcal{G}} \vdash_\varphi \psi \Leftrightarrow \vee_{\{s \in S^1 : \psi \in s\}} \varphi_s.$$

(Here we need the observation that by C1 and RC1, nothing is lost by our assumption that $C_G\psi \in s$ iff $E_G(\psi \wedge C_G\psi) \in s$.) Negating both sides of $\Leftrightarrow$, we get

$$\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \neg\psi \Leftrightarrow \wedge_{\{s \in S^1 : \psi \in s\}} \neg\varphi_s. \tag{2}$$

Thus, if $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \neg\varphi_s$ for each set s containing $\neg\varphi$, it follows by standard propositional reasoning that $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \varphi$, as desired.

While this general approach to proving completeness is quite standard, we must take extra care because of our insistence on restricting to symbols that appear in φ, particularly when dealing with the case when a state seems inconsistent due to a formula of the form $\neg E_G\psi$ or $\neg C_G\psi$ not being satisfied. This is where the axioms E1 and E2 come into play.

To prove (1), we first need a number of basic facts of epistemic logic and some preliminary observations. The basic facts (which are easily proved using Prop, E3 (or K1 when $G = \{i\}$), MP, and EGen (or KGen); see [FHMV95, p. 51, 94]) are that if ψ and ψ' involve only modal operators in φ, then

$$\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} E_G(\psi \wedge \psi') \Leftrightarrow E_G\psi \wedge E_G\psi' \tag{3}$$

and

$$\text{if } \mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \psi \Rightarrow \psi' \text{ then } \mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} E_G\psi \Rightarrow E_G\psi'. \tag{4}$$

Assume by induction that for all $s \in S^1 - S^j$, we have $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \neg\varphi_s$. We now show that if $s \in S^j$ does not seem consistent then $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \neg\varphi_s$, by considering in turn each of the two ways s may seem inconsistent.

First suppose that s does not seem consistent because $\neg E_G\psi \in s$ and there is no state $t \in S^j$ such that $(s,t) \in \cup_{i \in G}\mathcal{K}_i$ and $\neg\psi \in t$. We show that

$$\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \varphi_s \Rightarrow E_G\psi. \tag{5}$$

Since $\neg E_G\psi$ is a conjunct of φ_s (since $\neg E_G\psi \in s$, by assumption), (5) shows that φ_s is $\mathrm{K}_{\mathcal{G}}^{C}$-inconsistent, as desired.

To prove (5), we first show that if $G \in \mathcal{G}_\varphi$, then

$$\text{if } (s,t) \notin \cup_{i \in G}\mathcal{K}_i, \text{ then } \mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \varphi_s \Rightarrow E_G\neg\varphi_t. \tag{6}$$

To prove (6), suppose that $(s,t) \notin \cup_{i \in G}\mathcal{K}_i$. For each $i \in G$, there must be some $G^{i,t} \in \mathcal{G}_\varphi$ and formula $E_{G^{i,t}}\theta$ such that $i \in G$, $E_{G^{i,t}}\theta \in s$ and $\neg\theta \in t$. Since $E_{G^{i,t}}\theta \in s$ and $\neg\theta \in t$ it is immediate that $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \varphi_s \Rightarrow E_{G^{i,t}}\theta$ and $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \theta \Rightarrow \neg\varphi_t$. Now applying (4) and propositional reasoning, we get that $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \varphi_s \Rightarrow E_{G^{i,t}}\neg\varphi_t$. Since we can find such a $G^{i,t}$ for each $i \in G$, we have that $G \subseteq \cup_{i \in G} G^{i,t}$. Since G is finite, by E2, we have $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \varphi_s \Rightarrow E_G\neg\varphi_t$, as desired.

Returning to the proof of (5), note that (since $E_G\psi \in s$) if $\neg\psi \in t$ then $(s,t) \notin \cup_{i \in G}\mathcal{K}_i$. Thus, from (6) and (3), we have

$$\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \varphi_s \Rightarrow E_G(\wedge_{\{t \in S^j : \neg\psi \in t\}} \neg\varphi_t). \tag{7}$$

By the induction hypothesis, for all states in $t \in S^1 - S^j$, we have that $\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \neg\varphi_t$. Thus, using (2), we have

$$\mathrm{K}_{\mathcal{G}}^{C} \vdash_{\varphi} \psi \Leftrightarrow \wedge_{\{t \in S^j : \neg\psi \in t\}} \neg\varphi_t. \tag{8}$$

(5) now follows from (4), (7), and (8).

Finally, we must show that if $\neg C_G\psi \in s$ and there is no state $t \in S^j$ G-reachable from s in S^j such that $\neg\psi \in t$, then $\mathrm{K}_{\mathcal{G}}^C \vdash_\varphi \varphi_s \Rightarrow C_G\psi$, again showing that φ_s is $\mathrm{K}_{\mathcal{G}}^C$-inconsistent. This follows by a relatively straightforward modification of the completeness proof given in [FHMV95, HM92], so we just sketch the details here. Let $T_1 = \{t \in S^j : \neg C_G\psi \in t$ and there is no state $t' \in S^j$ G-reachable from t in S^j such that $\neg\psi \in t'\}$ and $T_2 = \{t \in S^j : C_G\psi \in t\}$. Let T_i' consist of those states in T_i that also contain ψ, $i = 1, 2$. Let $T = T_1 \cup T_2$ and let $T' = T_1' \cup T_2'$. We claim that there is no pair $(t, t') \in \cup_{i\in G}\mathcal{K}_i$ such that $t \in T$ and $t' \in S^j - T'$. It is immediate that if $t \in T_2$ then (since $\psi \wedge C_G\psi \in t/E_G \subseteq t'$) $t' \in T_2'$. If $t \in T_1$ and $t' \in S^j - T'$, then either $\neg\psi \in t'$ or $\neg C_G\psi \in t'$ and there is a state t'' G-reachable from t' in S^j such that $\neg\psi \in t''$. This means that either t' or t'' is a state G-reachable from t in S^j containing $\neg\psi$. This contradicts the fact that $t \in T_1$.

It now follows from (6) that for all $t \in T$ and $t' \in S^j - T'$, we have

$$\mathrm{K}_{\mathcal{G}}^C \vdash_\varphi \varphi_t \Rightarrow E_G\neg\varphi_{t'}. \tag{9}$$

Let $\varphi_T = \vee_{t\in T}\varphi_t$ and let $\varphi_{T'} = \vee_{t'\in T'}\varphi_{t'}$. By propositional reasoning, we have $\mathrm{K}_{\mathcal{G}}^C \vdash_\varphi \varphi_{T'} \Leftrightarrow (\varphi_T \wedge \psi)$. It easily follows from (3), (4), and (9) that $\mathrm{K}_{\mathcal{G}}^C \vdash_\varphi \varphi_t \Rightarrow E_G\varphi_{T'}$. Since this is true for all $t \in T$, we have

$$\mathrm{K}_{\mathcal{G}}^C \vdash_\varphi \varphi_T \Rightarrow E_G(\varphi_T \wedge \psi). \tag{10}$$

By applying RC1 and the fact that $s \in T$, we have $\mathrm{K}_{\mathcal{G}}^C \vdash_\varphi \varphi_s \Rightarrow C_G\psi$. Since $\neg C_G\psi \in s$, it follows that φ_s is $\mathrm{K}_{\mathcal{G}}^C$-inconsistent.

This completes the completeness proof in the case of $\mathcal{M}_{\mathcal{A}}$. To deal with $\mathcal{M}_{\mathcal{A}}^r$, we must just show that if s is eliminated because $(s, s) \notin \mathcal{K}_i$ for some $i \in \mathcal{A}$, then $\mathrm{T}_{\mathcal{G}}^C \vdash_\varphi \neg\varphi_s$; all other cases are identical. But if $(s, s) \notin \mathcal{K}_i$, then there must be some G and ψ such that $i \in G$, $E_G\psi \in s$, and $\neg\psi \in s$. Since $(\mathrm{T}_{\mathcal{G}}^C)^+$ includes the axiom $E_G\psi \Rightarrow \psi$, we have that $(\mathrm{T}_{\mathcal{G}}^C)^+ \vdash_\varphi \neg\varphi_s$, as desired. ∎

Proof of Theorem 3.1 for $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}^r$: We have already observed that the axioms are sound. For completeness, suppose that φ is valid with respect to $\mathcal{M}_{\mathcal{A}}$. By Proposition 4.1, so is φ^{σ_1}. By Lemma 4.7, there is a proof of φ^{σ_1} in $\mathrm{K}_{\mathcal{G}_\varphi}^C$ that mentions only the modal operators in φ^{σ_1}. Given a formula ψ in which the only modal operators that appear are modal operators that appear in φ^{σ_1} (and thus have the form $E_{\sigma_1(G)}$, $C_{\sigma_1(G)}$, and $K_{\sigma_1(i)}$, for sets G and $\{i\}$ in $\mathcal{G}_\varphi$) let ψ^{τ_1} be the unique formula all of whose modal operators appear in φ such that $(\psi^{\tau_1})^{\sigma_1} = \psi$. Lemma 4.6 assures us that ψ^{τ_1} is well defined. We can pull the proof of φ^{σ_1} back to a proof of φ, by replacing each occurrence of a formula ψ in the proof by ψ^{τ_1}.

The argument for $\mathcal{M}_{\mathcal{A}}^r$ is identical, except that the proof uses instances of the axiom E5. These can be eliminated by using E1 and K2, as we observed earlier (although now the proof of φ may use modal operators K_i that do not appear in φ). ∎

4.2 Dealing with $\mathcal{M}_{\mathcal{A}}^{rt}$

Proposition 4.1 as it stands does not hold for $\mathcal{M}_{\mathcal{A}}^{rt}$. There is no guarantee that the translated formula is satisfiable in $\mathcal{M}_{\mathcal{A}}^{rt}$, even if φ is. Indeed, suppose that $\mathcal{G}$ is closed under intersection and complementation, so that we can use the function σ of Proposition 3.5. Suppose that φ is the formula $E_G p \wedge \neg E_G E_G p$, where $|G| \geq 2$. The formula φ^σ looks syntactically identical, except that $\sigma(G)$ is a single agent in $\mathcal{A}'$. We cannot make the $\mathcal{K}_G$ relation transitive and still satisfy φ^σ. More generally, to deal with $\mathcal{M}_{\mathcal{A}}^{rt}$, we must be careful in how we deal with singleton sets.

As a first step, we define *mixed* structures. Since we also need these to deal with $\mathcal{M}_{\mathcal{A}}^{rst}$ and $\mathcal{M}_{\mathcal{A}}^{elt}$, we define three types of mixed structures at once. We say that a binary relation $\mathcal{K}$ is *secondarily reflexive* [Che80] if $(s,t) \in \mathcal{K}$ implies $(t,t) \in \mathcal{K}$. Let $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rt}$ (resp., $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rst}$; $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{elt}$) consist of structures $M = (S, \pi, \{\mathcal{K}_i : i \in \mathcal{A}_1 \cup \mathcal{A}_2\})$ where the relations $\mathcal{K}_i$ for $i \in \mathcal{A}_1$ are reflexive and transitive (resp., reflexive, symmetric and transitive; Euclidean, serial and transitive) and the relation $\mathcal{K}_i$ for $i \in \mathcal{A}_2$ are reflexive (resp., reflexive and symmetric; serial and secondarily reflexive).

We can now define our translation in the case of $\mathcal{M}_{\mathcal{A}}^{rt}$. Although we can in fact get an analogue to Proposition 4.1 for $\mathcal{M}_{\mathcal{A}}^{rt}$, it turns out to be easier to provide a translation that combines Proposition 4.1 and Lemma 4.4, rather than separating them. As suggested by Proposition 3.3, the translation involves $\mathcal{R}(\mathcal{G}_\varphi^1)$, rather than $\mathcal{R}(\mathcal{G}_\varphi)$. Given a formula φ, let $\mathcal{A}^{\varphi,rt} = \{\mathcal{H} : \exists G[(G,\mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi^1)]\}$. Let $\mathcal{A}_1 = \{\mathcal{H} : \exists G[(G,\mathcal{H}) \in \mathcal{R}(G_\varphi^1), |G - \cup\mathcal{H}| = 1]\}$; let $\mathcal{A}_2 = \mathcal{A}^{\varphi,rt} - \mathcal{A}_1$. Define $\sigma_2 : \mathcal{A} \to \mathcal{A}^{\varphi,rt}$ as before: $\sigma_2(i) = \mathcal{H}$ if $i \in A_\mathcal{H}$ and $\sigma_2(i)$ is undefined otherwise. Given $\mathcal{H} \in \mathcal{A}^{\varphi,rt}$, we define $\tau_2(\mathcal{H}) = \cap(\mathcal{G}_\varphi^1 - \mathcal{H})$. Since it is easy to see that $\mathcal{R}(\mathcal{G}_\varphi^1) = \mathcal{R}(\mathcal{G}_\psi)$ for some appropriate ψ, it is immediate that Lemma 4.4 applies to σ_2 and τ_2.

Proposition 4.8: *φ is satisfiable in $\mathcal{M}_{\mathcal{A}}^{rt}$ iff φ^{σ_2} is satisfiable in $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rt}$.*

Proof: First suppose that $(M,s) \models \varphi$, where $M \in \mathcal{M}_{\mathcal{A}}^{rt}$. We convert $M = (S, \pi, \{\mathcal{K}_i : i \in \mathcal{A}\})$ into a structure $M' = (S, \pi, \{\mathcal{K}_\mathcal{H} : \mathcal{H} \in \mathcal{A}^{\varphi,rt}\})$ as before, by defining $\mathcal{K}_\mathcal{H} = \cup\{\mathcal{K}_i : i \in \tau_2(\mathcal{H})\}$. Since Lemma 4.4 applies, the proof that $(M',s) \models \varphi$ is identical to that in Proposition 4.1. We must only show that $M' \in \mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rt}$. Since the union of reflexive relations is reflexive, it is immediate that $\mathcal{K}_\mathcal{H}$ is reflexive for $\mathcal{H} \in \mathcal{A}_2$. If $\mathcal{H} \in \mathcal{A}_1$, then $|A_\mathcal{H}| = 1$. Suppose that $A_\mathcal{H} = \{i\}$. We claim that $\tau_2(\mathcal{H}) = \{i\}$. By construction, $\{i\} \in \mathcal{G}_\varphi^1$. We cannot have $\{i\} \in \mathcal{H}$, since $i \notin \cup\mathcal{H}$. Thus $\{i\} \in \mathcal{G}_\varphi^1 - \mathcal{H}$, so $\tau_2(\mathcal{H}) = \cap(\mathcal{G}_\varphi^1 - \mathcal{H}) \subseteq \{i\}$. Since $\tau_2(\mathcal{H}) \neq \emptyset$ by Lemma 4.4(c), we must have $\tau_2(\mathcal{H}) = \{i\}$. Thus, $\mathcal{K}_\mathcal{H} = \mathcal{K}_i$, so $\mathcal{K}_\mathcal{H}$ is reflexive and transitive.

For the opposite direction we need to work a little harder than before, because we must ensure that all the $\mathcal{K}_i$ relations are reflexive and transitive for all $i \in \mathcal{A}$. Supppose $(M,s) \models \varphi^{\sigma_2}$ for some $M = (S, \pi, \{\mathcal{K}_\mathcal{H} : \mathcal{H} \in \mathcal{A}^{\varphi,rt}\}) \in \mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rt}$. Let S_0 and S_1 be two disjoint copies of S. For a state $s \in S$, let s_i be the copy of s in S_i, $i = 0,1$. Let $M' = (S', \pi', \{\mathcal{K}_i : i \in \mathcal{A}\})$ be defined as follows:

- $S' = S_0 \cup S_1$.
- $\pi'(s_i) = \pi(s)$ for $i = 0,1$.
- If $\sigma_2(i) \in \mathcal{A}_1$, define $\mathcal{K}_i = \{(s_i,t_j) : (s,t) \in \mathcal{K}_{\sigma_2(i)}, i,j \in \{0,1\}\}$. $\mathcal{K}_i$ is clearly reflexive and transitive in this case, since $\mathcal{K}_{\sigma_2(i)}$ is.

- If $\sigma_2(i) = \mathcal{H} \in \mathcal{A}_2$, note that $|A_{\mathcal{H}}| \geq 2$. It is immediate from the definition that $\sigma_2(i) = \mathcal{H}$ for all $i \in A_{\mathcal{H}}$. Pick some $i_{\mathcal{H}} \in A_{\mathcal{H}}$. If $i = i_{\mathcal{H}}$, then define $\mathcal{K}_i = \{(s_0, t_1) : (s,t) \in \mathcal{K}_{\mathcal{H}}\} \cup \{(s_j, s_j) : j \in \{0,1\}\}$; if $i \neq i_{\mathcal{H}}$, define $\mathcal{K}_i = \{(s_1, t_0) : (s,t) \in \mathcal{K}_{\mathcal{H}}\} \cup \{(s_j, s_j) : j \in \{0,1\}\}$. Clearly $\mathcal{K}_i$ is reflexive and transitive.

This construction guarantees that

$$(s,t) \in \mathcal{K}_{\mathcal{H}} \text{ iff } (s_0,t_1),(s_1,t_0) \in \cup_{\{i:\sigma_2(i)=\mathcal{H}\}}\mathcal{K}_i \tag{11}$$

and

$$(s_1,t_0) \in \cup_{\{i:\sigma_2(i)=\mathcal{H}\}}\mathcal{K}_i \text{ iff } (s_0,t_1) \in \cup_{\{i:\sigma_2(i)=\mathcal{H}\}}\mathcal{K}_i. \tag{12}$$

A straightforward argument by induction on structure now shows that if $\psi \in \mathcal{L}^C_{\mathcal{G}^1_\varphi}$, then the following are equivalent for all $t \in S$:

- $(M,t) \models \psi^{\sigma_2}$,
- both $(M',t_0) \models \psi$ and $(M',t_1) \models \psi$,
- $(M',t_0) \models \psi$ or $(M',t_1) \models \psi$.

Of course, the interesting cases are if ψ is of the form $K_i\psi'$, $E_G\psi'$, or $C_G\psi'$. These follow immediately from observations (11) and (12). ∎

The next step is to get an analogue of Theorem 4.5 for $\mathcal{M}^{rt}_{\mathcal{A}_1+\mathcal{A}_2}$. The basic idea of the proof is the same as that of Theorem 4.5. However, in our construction, we need to make the $\mathcal{K}_i$ relations transitive. To see the difficulty, suppose that φ is $K_1 p \wedge E_G q$, where G is a set of agents containing 1. Recall that in Theorem 4.5, states are consistent subsets of $Sub^+(\varphi)$. Let s, t, and u be states such that $s = \{K_1p, E_Gq, p, q\}$, $t = \{K_1p, \neg E_Gq, p, q\}$, and $u = \{K_1p, \neg E_Gq, p, \neg q\}$. With our previous construction, we would have both $(s,t) \in \mathcal{K}_1$ and $(t,u) \in \mathcal{K}_1$. By transitivity, we should also have $(s,u) \in \mathcal{K}_1$. But since $E_Gq \in s$ and $\neg q \in u$, we have $(s,u) \notin \mathcal{K}_1$. Nevertheless, each of s, t, and u individually seems consistent. Which state should we eliminate in order to preserve transitivity?

To deal with this problem, we need to put more information (i.e., more formulas) into each state. Intuitively, if $(s,t) \in \mathcal{K}_i$, then we should have $K_iq \in t$, because if $E_Gq \in s$, then K_iq should also be in s, as should K_iK_iq by K4. It would then follow that K_iq should be in t. This, in turn, would guarantee that $(t,u) \notin \mathcal{K}_i$, since $q \notin u$.

What we would like to do now is to augment $Sub(\varphi)$ by including all formulas $K_i\psi$ such that $E_G\psi \in Sub(\varphi)$ and $i \in G \cap \mathcal{A}_1$. (We restrict to $\mathcal{A}_1$ since these are the only relations that are required to be transitive.) While this approach can be used to force the $\mathcal{K}_i$ relations to be transitive, the resulting set of formulas can have size $O(|\mathcal{A}_1||\varphi|)$, which means the resulting state space (the analogue of S^1) could then have size $2^{|\mathcal{A}_1||\varphi|}$. This would not give us the desired complexity bounds. Thus, we must proceed a little more cautiously.

Theorem 4.9: *If $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is finite and there is an algorithm for deciding if $i \in G$ for $G \in \mathcal{G}$ that runs in time linear in $|\mathcal{A}|$, then there is a constant $c > 0$ (independent of $\mathcal{A}$) and an algorithm that, given a formula φ of $\mathcal{L}^C_{\mathcal{G}}$, decides if φ is satisfiable in $\mathcal{M}^{rt}_{\mathcal{A}_1+\mathcal{A}_2}$ and runs in time $O(|\mathcal{A}|2^{c|\varphi|})$.*

Proof: We assume for ease of exposition that $\mathcal{A}_1 \neq \emptyset$; we leave the straightforward modification in case $\mathcal{A}_1 = \emptyset$ to the reader. For each $i \in \mathcal{A}_1$, let $ESub_i(\varphi)$ be the least set containing $Sub(\varphi)$ such that if $E_G\varphi \in Sub(\varphi)$ and $i \in G$, then $K_i\varphi \in ESub_i(\varphi)$. It is easy to see that $|ESub_i(\varphi)| \leq 2|Sub(\varphi)|$, since we add at most one formula for each formula in $|Sub(\varphi)|$. Let S_i^1 consist of all the subsets of $ESub_i^+(\varphi)$ that are maximally consistent, and now let $S^1 = \cup_{i\in\mathcal{A}_1} S_i^1$. Note that, as modified, $|S^1| \leq 2^{2|\varphi|}$. Thus, this modification keeps us safely within the desired exponential time bounds.

We keep the definition of $\mathcal{K}_i$ unchanged for $i \in \mathcal{A}_2$ (i.e., $(s,t) \in \mathcal{K}_i$ iff $s/\overline{K_i} \subseteq t$), but we need to modify it for $i \in \mathcal{A}_1$. We redefine $\mathcal{K}_i$ for $i \in \mathcal{A}_1$ by defining $(s,t) \in \mathcal{K}_i$ iff $s/\overline{K_i} \cup \{K_i\psi : K_i\psi \in s\} \subseteq t \cap (t/\overline{K_i} \cup \{K_i\psi : K_i\psi \in t\})$. It is easy to check that this modification forces the $\mathcal{K}_i$ relations to be transitive. We force all the $\mathcal{K}_i$ relations to be reflexive just as with $\mathcal{M}^r_{\mathcal{A}}$, by eliminating $s \in S^1$ if $(s,s) \notin \mathcal{K}_i$ for some $i \in \mathcal{A}_1 \cup \mathcal{A}_2$. The remainder of the construction—eliminating the states that do not seem consistent—is unchanged.

We now need to show that the algorithm is correct. First suppose that φ is satisfiable in $\mathcal{M}^{rt}_{\mathcal{A}_1+\mathcal{A}_2}$. In that case, $(M, s_0) \models \varphi$ for some structure $M = (S, \pi, \{\mathcal{K}'_i : i \in \mathcal{A}_1 \cup \mathcal{A}_2\}) \in \mathcal{M}^{rt}_{\mathcal{A}_1+\mathcal{A}_2}$. We can associate with each state $s \in S$ and $i \in \mathcal{A}_1$ the state s_i^* in S_i^1 consisting of all the formulas $\psi \in ESub_i(\varphi)$ such that $(M,s) \models \psi$. It is easy to see that if $(s,t) \in \mathcal{K}'_i$ then $(s_j^*, t_i^*) \in \mathcal{K}_i$ for all j.[3] Using this observation, a straightforward induction shows that the states s_i^* for $s \in S$ always seem consistent, and thus are in S^j for all j and all $i \in \mathcal{A}_1$. Moreover, $\varphi \in (s_0)_i^*$ for all $i \in \mathcal{A}_1$. Thus, the algorithm will declare that φ is satisfiable, as desired.

Conversely, suppose that the algorithm declares that φ is satisfiable. We construct a structure $M = (S, \pi, \{\mathcal{K}'_i : i \in \mathcal{A}_1 \cup \mathcal{A}_2\}) \in \mathcal{M}^{rt}_{\mathcal{A}_1+\mathcal{A}_2}$ in which φ is satisfied just as Theorem 4.5. Our modified construction guarantees that the $\mathcal{K}'_i$ relations are all reflexive and the ones in $\mathcal{A}_1$ are transitive. ∎

We are almost ready to prove Theorem 3.4 for $\mathcal{M}^{rt}_{\mathcal{A}}$. However, we first we need to characterize the complexity of translating from φ to φ^{σ_2}. In particular, we need a bound on the number of elements in $\mathcal{R}(\mathcal{G}^1_\varphi)$ and the number of oracle calls required to compute them. To do this, we first define two auxiliary sequences of sets $\mathcal{D}_i^m(\mathcal{J})$ and $\mathcal{E}_i^m(\mathcal{J})$, $i = 1, 2, 3, \ldots$. (We omit the parenthetical $\mathcal{J}$ when it is clear from context.) Fix m. Let $\mathcal{D}_0^m = \mathcal{J}$ and $\mathcal{D}_{i+1}^m = \mathcal{J} \cup \{G - \cup\mathcal{H} : (G, \mathcal{H}) \in \mathcal{R}(\mathcal{D}_i^m) \text{ and } |G - \cup\mathcal{H}| \leq m\}$; let $\mathcal{E}_i^m = \mathcal{D}_i^m - \mathcal{D}_0^m$. Set $\mathcal{D}^m = \cup_i \mathcal{D}_i^m$ and $\mathcal{E}^m = \cup_i \mathcal{E}_i^m$. Finally, denote $\mathcal{R}(\mathcal{D}^m)$ by $\mathcal{R}^m(\mathcal{J})$. It is easy to check that $\mathcal{D}_0^m \subseteq \mathcal{D}_1^m \subseteq \ldots$ and that $\mathcal{R}^m(\mathcal{J}) = \cup_i \mathcal{R}(\mathcal{D}_i^m)$.

The next lemma provides partial motivation for these definitions.

Lemma 4.10: $\mathcal{R}^m(\mathcal{J}) = \mathcal{R}(\mathcal{J}^m)$.

Proof: An easy induction on i shows that $\mathcal{D}_i^m(\mathcal{J}) \subseteq \mathcal{J}_i^m$ (as defined in Definition 3.2) for all i, so $\mathcal{D}^m \subseteq \mathcal{J}^m$. We next show that every set in $\mathcal{J}_i^m$ is the union of sets in $\mathcal{D}^m$, by induction on i. This is immediate if $i = 0$, since $\mathcal{J}_0^m = \mathcal{D}_0^m = \mathcal{J}$. Suppose that the result holds for $\mathcal{J}_i^m$; we show it for $\mathcal{J}_{i+1}^m$. Suppose that $H \in \mathcal{J}_{i+1}^m$. If $H \in \mathcal{J}$, then clearly $H \in \mathcal{D}^m$. Thus, without loss of generality, $H \in \mathcal{J}_{i+1}^m - \mathcal{J}$, which means that $|H| \leq m$. Let H' be the union of all sets

[3]Note that it is not necessarily the case that $(s_j^*, t_{j'}^*) \in \mathcal{K}_i$ for $j' \neq i$. For example, suppose φ is the formula E_Gp, $i \in G \cap \mathcal{A}_1$, and M is such that $(M,s) \models E_Gp \wedge p$, $(M,t) \models \neg E_Gp \wedge p$, and $(s,t) \in \mathcal{K}_i$. Then for $i \neq j, j'$ and $j \notin G$, we have $s_j^* = \{E_Gp, p\}$ and $t_{j'}^* = \{p, \neg E_Gp\}$. Since $p \in s_j^*/\overline{K_i} - t_{j'}^*/\overline{K_i}$, we have that $(s_j^*, t_{j'}^*) \notin \mathcal{K}_i$.

in $\mathcal{D}^m$ contained in H. If $H' = H$, then we are done. Suppose by way of contradiction that $H - H' \neq \emptyset$. We obtain a contradiction to the choice of H' by showing that $H - H'$ contains a set in $\mathcal{D}^m$.

Since H' is finite, it can be written as a finite union of sets in $\mathcal{D}^m$, say of $\mathcal{H}_1 = H_1, \ldots, H_k$. Since $H \in \mathcal{J}_{i+1}^m - \mathcal{J}$, $H = G - \cup\mathcal{H}_2$ for some $G \in \mathcal{J}$ and $\mathcal{H}_2 \subseteq \mathcal{J}_i^m$. By the induction hypothesis, there exists some $\mathcal{H}_3 \subseteq \mathcal{D}^m$ such that $\cup\mathcal{H}_2 = \cup\mathcal{H}_3$. There must exist some set $\mathcal{H}_4 \supseteq \mathcal{H}_1 \cup \mathcal{H}_3$ such that $(G, \mathcal{H}_4) \in \mathcal{R}(\mathcal{D}^m)$. But then $H - H' \supseteq G - \cup\mathcal{H}_4 \in \mathcal{D}^m$, and we obtain the desired contradiction.

It now easily follows that $\mathcal{R}(\mathcal{J}^m) = \mathcal{R}(\mathcal{D}^m) = \mathcal{R}^m(\mathcal{J})$. ∎

The following result will be used to help compute the elements of $\mathcal{R}^m(\mathcal{J})$.

Lemma 4.11: *Let $\mathcal{J}$ be a set of subsets of $\mathcal{A}$ with $|\mathcal{J}| = n$.*

(a) If $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{D})$, where $\mathcal{J} \subseteq \mathcal{D} \subseteq \mathcal{J}^*$, then $G - \cup\mathcal{H}$ is an atom over $\mathcal{J}$.

(b) $\mathcal{J} \subseteq \mathcal{D}_i^m \subseteq \mathcal{J}^*$ for all i, m.

(c) $|\{\mathcal{H} : \exists G \in \mathcal{D}^m((G, \mathcal{H}) \in \mathcal{R}^m(\mathcal{J})\}| \leq 2^n$.

(d) If $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{D}_i^m)$, then either $G \in \mathcal{J}$ and $\mathcal{E}_i^m \subseteq \mathcal{H}$ or $A_{\mathcal{H}} \in \mathcal{E}_i^m$ and $\mathcal{E}_i^m - \{A_{\mathcal{H}}\} \subseteq \mathcal{H}$. Moreover, if $(G, \mathcal{H}) \in \mathcal{R}^m(\mathcal{J})$, then either $G \in \mathcal{J}$ and $\mathcal{E}^m \subseteq \mathcal{H}$ or $(G - \cup\mathcal{H}) \in \mathcal{E}^m$ and $\mathcal{E}^m - \{G - \cup\mathcal{H}\} \subseteq \mathcal{H}$.

(e) $\mathcal{D}^m = \mathcal{D}_n^m$ and $\mathcal{E}^m = \mathcal{E}_n^m$.

Proof: For part (a), we know from Lemma 4.3(a) that if $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{D})$, then $G - \cup\mathcal{H}$ is an atom over $\mathcal{D}$. Since $\mathcal{J} \subseteq \mathcal{D} \subseteq \mathcal{J}^*$, it is immediate that it must in fact be an atom over $\mathcal{J}$ as well. (Recall that $\mathcal{J}^*$ is the algebra generated by $\mathcal{J}$.)

Part (b) follows immediately from (a), since an easy induction on i shows that $\mathcal{E}_i^m \subseteq \mathcal{J}^*$.

For part (c), by Lemma 4.3(a), it follows that $A_{\mathcal{H}}$ is an atom over $\mathcal{D}^m$. But since $\mathcal{J} \subseteq \mathcal{D}^m = \cup_i \mathcal{D}_i^m \subseteq \mathcal{J}^*$ by part (b), it follows that $A_{\mathcal{H}}$ is actually at atom over $\mathcal{J}$. Moreover if $(G', \mathcal{H}') \in \mathcal{R}^m(\mathcal{J})$ and $\mathcal{H} \neq \mathcal{H}'$, then it follows from Lemma 4.3(c) that $A_{\mathcal{H}} \neq A_{\mathcal{H}'}$. Since there are at most 2^n atoms over $\mathcal{J}$, part (c) follows.

For part (d), if $(G, \mathcal{H}) \in \mathcal{R}(D_i^m)$ then, by Lemma 4.3(a), $A_{\mathcal{H}} = G - \cup\mathcal{H}$ is an atom over $\mathcal{D}_i^m$ and has the form $\cap(\mathcal{D}_i^m - \mathcal{H}) \cap \cap\{\overline{H} : H \in \mathcal{H}\}$. By the arguments of part (c), $A_{\mathcal{H}}$ is also an atom over $\mathcal{J}$. We say that the sets in $\mathcal{D}_i^m - \mathcal{H}$ *appear positively* in $A_{\mathcal{H}}$ and the sets in $\mathcal{H}$ *appear negatively* in $A_{\mathcal{H}}$. If one of the sets $G' \in \mathcal{E}_i^m$ appears positively in $A_{\mathcal{H}}$ then clearly $A_{\mathcal{H}} \subseteq G'$. But since the elements of $\mathcal{E}_i^m$ are also atoms over $\mathcal{J}$, it follows that in this case $A_{\mathcal{H}} = G' \in \mathcal{E}_i^m$ and, since $\mathcal{H}$ is G-maximal, $\mathcal{E}_i^m - \{A_{\mathcal{H}}\} \subseteq \mathcal{H}$. Otherwise, $\mathcal{E}_i^m \subseteq \mathcal{H}$ as required; moreover, since $\mathcal{D}_i^m = \mathcal{E}_i^m \cup \mathcal{J}$ and $G \notin \mathcal{H}$, we must have $G \in \mathcal{J}$. The argument for the second half of (d) is identical.

Clearly the two claims in part (e) are equivalent. We prove the second. As observed in the proof of (c), every set in $\mathcal{E}^m$ is an atom A over $\mathcal{J}$. It is easy to see that there are no atoms in $\mathcal{E}^m$ where all n sets in $\mathcal{J}$ appear negatively, since every set in $\mathcal{E}^m$ is a nonempty subset of some

$G \in \mathcal{J}$. (This can be proved by induction on i for each $\mathcal{E}_i^m$.) We prove by induction on i that if $A \in \mathcal{E}^m$ and $n-i$ sets appear negatively in A for $i \geq 1$, then $A \in \mathcal{E}_i^m$.

Clearly if $i = 1$, then $A = G - (H_1 \cup \ldots \cup H_{n-1})$, and $\mathcal{H} = \{H_1, \ldots, H_{n-1}\}$ is a G-maximal subset of $\mathcal{J}$. Thus, $(G, \mathcal{H}) \in \mathcal{D}_1^m$ and $A \in \mathcal{E}_1^m$. Suppose that the result is true if $i = k$ and suppose that $n-(k+1)$ sets appear negatively in A. As $A \in \mathcal{E}^m$, there must be some minimal j such that $A \in \mathcal{E}_{j+1}^m$. By definition, $A = \mathcal{A}_H$ for some $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{D}_j^m)$. By (d), either $A = G - (\cup\mathcal{H}' \cup \mathcal{E}_j^m)$ and $\mathcal{H}' \subseteq \mathcal{J}$ or $A \in \mathcal{E}_j^m$. The latter case contradicts our choice of m, so we may assume that $A = G - (\cup\mathcal{H}' \cup \mathcal{E}_j^m)$ and $\mathcal{H}' \subseteq \mathcal{J}$. It is easy to see that $\mathcal{H}'$ must consist of precisely the sets in $\mathcal{J}$ that appear negatively in A. (If it did not include all the sets that appear negatively in A then $\mathcal{H}' \cup \mathcal{E}_j^m$ would not be a G-maximal subset of $\mathcal{J} \cup \mathcal{E}_j^m$; if it includes any sets that appear postively then A would be empty.) Let $\mathcal{E}'$ consist of all the atoms A' in $\mathcal{E}_j^m$ in which the set of sets in $\mathcal{J}$ that appear negatively in A' is a strict superset of $\mathcal{H}'$. It is easy to see that $G - (\cup\mathcal{H}' \cup \mathcal{E}_j^m) = G - (\cup\mathcal{H}' \cup \mathcal{E}')$, since all the sets in $\mathcal{E}_j^m - \mathcal{E}'$ must be disjoint from $G - \cup\mathcal{H}'$. (This is clear for the $B \in \mathcal{E}_j^m - \mathcal{E}'$ for which some set appearing negatively in A does not appear negatively in B. On the other hand, if the same sets appear negatively in B as in A then $B = A$ and we contradict the minimality of j.) By the induction hypothesis, $\mathcal{E}' \subseteq \mathcal{E}_{n-k}^m$. Thus, $A = G - (\mathcal{H}' \cup \mathcal{E}_{n-k}^m) \in \mathcal{E}_{n-k+1}^m$, as desired. ∎

We remark that a simpler proof, just using the fact that there are at most 2^n atoms over $\mathcal{J}$, can be used to show that $\mathcal{E}_{n'}^m = \mathcal{E}_{2^n}^m$ for $n' > 2^n$. This simpler proof would suffice for the purposes of this subsection. However, we shall use the added information in (e) in Section 4.5.

Proof of Theorem 3.4 for $\mathcal{M}_{\mathcal{A}}^{rt}$: Again, the lower bound follows from standard results in [HM92].

For the upper bound, suppose that we are given a formula φ such that $n = |\varphi|$ and $\mathcal{H} \in \mathcal{A}^{\varphi,rt}$. By definition, there exists a G such that $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi^1)$. By Lemma 4.10, $\mathcal{R}^1(\mathcal{G}_\varphi) = \mathcal{R}(\mathcal{G}_\varphi^1)$. Thus, $\mathcal{H} \subseteq \mathcal{D}^1(\mathcal{G}_\varphi) = \mathcal{G}_\varphi \cup E_n^1(\mathcal{G}_\varphi)$. By Lemma 4.11(d), either $\mathcal{E}_n^1(\mathcal{G}_\varphi) \subseteq \mathcal{H}$ or $\mathcal{H}$ contains all but one element of $\mathcal{E}_n^1(\mathcal{G}_\varphi)$. Thus, we can uniquely characterize $\mathcal{H}$ by a pair $(\mathcal{H}', X)$, where $\mathcal{H}' = \mathcal{H} \cap \mathcal{G}_\varphi$ and $X = \mathcal{E}_n^1(\mathcal{G}_\varphi) - \mathcal{H}$ (so that X is either the empty set or a singleton). It should be clear that we can compute compute the set $\mathcal{E}_n^1(\mathcal{G}_\varphi)$ in time $O(n^2 2^{cn})$ and which of these (at most $2^{2n} + 2^n$) pairs is in $\mathcal{A}_1$ and $\mathcal{A}_2$ using at most $2n(2^{2n} + 2^n)$ calls to the oracle O_1.

By Lemmas 4.4(a) and 4.11, we can similarly compute the formula φ^{σ_2} in time $O(2^{cn})$ using $O(2^{cn})$ oracle calls. We now apply Proposition 4.8 and Theorem 4.9, just as we applied Proposition 4.1 and Theorem 4.5 in the case of $\mathcal{M}_{\mathcal{A}}$. ∎

We next want to prove Theorem 3.1 for $\mathcal{M}_{\mathcal{A}}^{rt}$. Just as with $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}^r$, we want to pull a proof of φ^{σ_2} back to a proof of σ. However, it is no longer true that we can necessarily prove φ^{σ_2} using only the modal operators that appear in φ^{σ_2}. We may also need to use $K_{\mathcal{H}}$ for $\mathcal{H} \in \mathcal{A}_1$. Fortunately, this does not cause us problems. The following extension of Lemma 4.6 is immediate.

Lemma 4.12: *The mapping σ_2 (when viewed as a map with domain $2^{\mathcal{A}}$) is injective on $\mathcal{G}_\varphi^1$.*

Let $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2}$ consist of the axioms in $(\mathrm{T}_{\mathcal{G}}^C)^+$ (so that, in particular, E5 is included), together with every instance of K4 ($K_i\varphi \Rightarrow K_iK_i\varphi$) for $i \in \mathcal{A}_1$. We write $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \psi$ if there is a proof of ψ in $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2}$ using only the modal operators that appear in φ and K_i for $i \in \mathcal{A}_1$.

Lemma 4.13: *If $\mathcal{A}$ is finite and $\varphi \in \mathcal{L}_{\mathcal{G}}^C$ is valid with respect to $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rt}$, then $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi$.*

Proof: The proof is similar to that of Lemma 4.7 for $\mathcal{M}_{\mathcal{A}}^r$, except that since the definition of the $\mathcal{K}_i$ relation is different, we must still check that the results still hold with the modified definition.

Suppose that $s \in S^j$ does not seem consistent because $\neg E_G\psi \in s$ and there is no state $t \in S^j$ such that $(s,t) \in \cup_{i\in G}\mathcal{K}_i$ and $\neg\psi \in t$. We want to show that $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow E_G\psi$. As before this suffices.

For each $i \in G$ and, by induction, each j, we have a provable equivalence for ψ similar to the one before: $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \psi \Leftrightarrow \wedge_{\{t\in S_i{}^j):\neg\psi\in t\}}\neg\varphi_t$. So it suffices to find, for each such i and each $t \in S_i^j$ with $\neg\psi \in t$, a $G^{i,t}$ containing i such that $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow E_{G^{i,t}}\neg\varphi_t$. For $i \in \mathcal{A}_2$, this follows just as before. For $i \in \mathcal{A}_1$, we show that $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\neg\varphi_t$. By our assumption $(s,t) \notin \mathcal{K}_i$. Thus, there exists some formula $\theta \in s/\overline{K_i} \cup \{K_i\theta : K_i\theta \in s\} - (t \cap (t/\overline{K_i} \cup \{K_i\theta : K_i\theta \in t\}))$. If $\theta \in s/\overline{K_i}$, then $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\theta$. If $\theta = K_i\theta'$ is in s, then $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\theta'$. By K4, we have that $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_iK_i\theta'$. Thus, in either case, we have $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\theta$. Since $\theta \in s/\overline{K_i} \cup \{K_i\theta : K_i\theta \in s\}$, it follows that $K_i\theta \in ESub_i(\neg\varphi)$. We cannot have $K_i\theta \in t$, for then (since $(t,t) \in \mathcal{K}_i$, so $t/\overline{K_i} \subseteq t$) we would have $\theta \in t \cap t/\overline{K_i}$, contradicting our choice of θ. Thus we must have that $\neg K_i\theta \in t$. It follows that $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi K_i\theta \Rightarrow \neg\varphi_t$. Using (4), we get that $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi K_iK_i\theta \Rightarrow K_i\neg\varphi_t$. Since $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi K_i\theta \Rightarrow K_iK_i\theta$ and, as shown earlier, $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\theta$, it follows that $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\neg\varphi_t$, as desired.

Finally, we must show that if $\neg C_G\psi \in s$ and there is no state $t \in S^j$ G-reachable from s in S^j such that $\neg\psi \in t$, then $\mathrm{S4}_{\mathcal{G}}^C \vdash_\varphi \varphi_s \Rightarrow C_G\psi$. This argument is identical to that given in the proof of Lemma 4.7, so we do not repeat it here. ∎

Proof of Theorem 3.1 for $\mathcal{M}_{\mathcal{A}}^{rt}$: Again, we have already observed that the axioms are sound. For completeness, suppose that φ is valid with respect to $\mathcal{M}_{\mathcal{A}}$. By Proposition 4.8, φ^{σ_2} is valid with respect to $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rt}$. By Lemma 4.13, there is a proof of φ^{σ_2} in $\mathrm{K}_{\mathcal{G}_\varphi}^C$ that mentions only the modal operators in φ^{σ_2} and the operators $K_{\mathcal{H}}$ for $\mathcal{H} \in \mathcal{A}_1$. Using Lemma 4.12, it follows that we can pull this back to a proof of φ in $\mathrm{S4}_{\mathcal{G}}^C$. ∎

4.3 Dealing with $\mathcal{M}_{\mathcal{A}}^{rst}$

$\mathcal{M}_{\mathcal{A}}^{rst}$ and $\mathcal{M}_{\mathcal{A}}^{elt}$ introduce additional complications. The translation used in Proposition 4.8 no longer suffices. We need to deal with the fact that in $\mathcal{M}_{\mathcal{A}}^{rst}$, we can test not only that whether a set is a singleton, but whether it has size k for any k. Given a formula φ, suppose that $|\varphi| = n$. We want to map $\mathcal{A}$ to a finite set of agents and prove an analogue of Propositions 4.8. The obvious analogue of $\mathcal{A}^{\varphi,rt}$ would be to consider the sets $\mathcal{H}$ such that $(G,\mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi^n)$. We essentially do this, except that we replace all sets of cardinality $\leq n$ by the singletons in them.

Given a set $\mathcal{J}$ of subsets of $\mathcal{A}$, let $\widetilde{\mathcal{J}}^m = \mathcal{D}^m(\mathcal{J}) \cup \{\{i\} : \exists G \in \mathcal{D}^m(\mathcal{J})(|G| \leq m,\ i \in G\}$. Let $\mathcal{A}^{\varphi,rst} = \{\mathcal{H} : \exists G((G,\mathcal{H}) \in \mathcal{R}(\widetilde{\mathcal{G}}^n_\varphi)\}$. Let $\mathcal{A}_1 = \{\mathcal{H} : \exists G[(G,\mathcal{H}) \in \mathcal{R}(\widetilde{\mathcal{G}}^n_\varphi),\ |G - \cup\mathcal{H}| = 1\}$; let $\mathcal{A}_2 = \mathcal{A}^{\varphi,rst} - \mathcal{A}_1$. Define $\sigma_3 : \mathcal{A} \to \mathcal{A}_1 \cup \mathcal{A}_2$ as before: $\sigma_3(i) = \mathcal{H}$ if $i \in A_\mathcal{H}$ and $\sigma_3(i)$ is undefined otherwise. Much as before, we define $\tau_3(\mathcal{H}) = \cap(\widetilde{\mathcal{G}}^n_\varphi - \mathcal{H})$. Since it is easy to see that $\mathcal{R}(\widetilde{\mathcal{G}}^n_\varphi) = \mathcal{R}(\mathcal{G}_\psi)$ for some appropriately chosen ψ, it is immediate that Lemma 4.4 applies without change to σ_3 and τ_3.

Lemma 4.14: *If $\mathcal{H} \in \mathcal{A}_2$, then $|A_\mathcal{H}| \geq n+1$.*

Proof: Suppose, by way of contradiction, that $\mathcal{H} \in \mathcal{A}_2$ and $1 \leq |A_\mathcal{H}| \leq n$. We must have $|A_\mathcal{H}| > 1$, for otherwise $\mathcal{H} \in \mathcal{A}_1$. Since $\mathcal{A}_2 \subseteq \mathcal{A}^{\varphi,rst}$, there must exist $G \in \mathcal{D}^n(\mathcal{G}_\varphi)$ such that $\mathcal{H}$ is G-maximal. But if $|A_\mathcal{H}| \leq n$, then every singleton subset of $A_\mathcal{H}$ is in $\widetilde{\mathcal{G}}^n_\varphi$. This contradicts the fact that $\mathcal{H}$ is G-maximal, because if $\mathcal{H}'$ is $\mathcal{H}$ together with one of these singleton subsets, we must have $G - \cup\mathcal{H}' \neq \emptyset$. ∎

Proposition 4.15: *φ is satisfiable in $\mathcal{M}^{rst}_\mathcal{A}$ iff φ^{σ_3} is satisfiable in $\mathcal{M}^{rst}_{\mathcal{A}_1+\mathcal{A}_2}$.*

Proof: First suppose that $(M,s) \models \varphi$, where $M \in \mathcal{M}^{rst}_\mathcal{A}$. We convert $M = (S,\pi,\{\mathcal{K}_i : i \in \mathcal{A}\})$ into a structure $M' = (S,\pi,\{\mathcal{K}_\mathcal{H} : \mathcal{H} \in \mathcal{A}^{\varphi,rst}\})$ as before, by defining $\mathcal{K}_\mathcal{H} = \cup\{\mathcal{K}_i : i \in \tau_3(A)\}$. As the union of symmetric relations is symmetric, the proof that this works is essentially identical to that in Lemma 4.8 for the case of $\mathcal{M}^{rt}_\mathcal{A}$.

For the opposite direction, suppose that $(M,s) \models \varphi^{\sigma_3}$ for some $M = (S,\pi,\{\mathcal{K}_\mathcal{H} : \mathcal{H} \in \mathcal{A}^{\varphi,rst}\}) \in \mathcal{M}^{rst}_{\mathcal{A}_1+\mathcal{A}_2}$. We must construct a structure $M' \in \mathcal{M}^{rst}_\mathcal{A}$ that satisfies φ. The state space for the structure M' will again consist of copies of S, but two copies no longer suffice to guarantee that the $\mathcal{K}_i$ relations are equivalence relations. In fact, we use countably many copies.

By Lemma 4.14, for each $\mathcal{H} \in \mathcal{A}_2$, there exist at least $n+1$ agents in $A_\mathcal{H}$. Choose $n+1$ such agents, and call them $i^0_\mathcal{H},\ldots,i^n_\mathcal{H}$. Partition $A_\mathcal{H}$ into $n+1$ disjoint sets $G_{\mathcal{H},j}$ with $i^j_\mathcal{H} \in G_{\mathcal{H},j}$. We build copies of M in a tree-like manner. We index the copies of M with strings of the form $((s_1,t_1),i_1,\ldots,(s_k,t_k),i_k)$, such that $s_j,t_j \in S$, i_j is $i^{j'}_\mathcal{H}$ for some $\mathcal{H} \in \mathcal{A}_2$ and $0 \leq j' \leq n$, $(s_j,t_j) \in \mathcal{K}_\mathcal{H}$, and $i_j \neq i_{j+1}$. Roughly speaking, between M_σ and $M_{\sigma\cdot((s_k,t_k),i_k)}$ we have edges for the $\mathcal{K}_i$ relations for $\{i\} \in \mathcal{A}_1$ and also edges between s_k and t_k in $\mathcal{K}_{i_k}$; however, there are no edges in $\mathcal{K}_j$ if $\{j\} \notin \mathcal{A}_1$ and $j \neq i_k$; moreover, there are no other edges in $\mathcal{K}_{i_k}$ except those required to assure reflexivity.

Before we can construct M', we need some preliminary observations. We can suppose that the states in S are numbered. Thus, for each state $s \in S$, if $(M,s) \models \neg C_G\psi$, there is a lexicographically minimal shortest path $(s_0,\ldots,s_k)$ such that $(s_i,s_{i+1}) \in \mathcal{K}_\mathcal{H}$ for some $\mathcal{H} \in G$ and $(M,s_k) \models \neg\psi$. Note that, for each $i \leq k$, $(M,s_i) \models \neg C_G\psi$ and $(s_i,\ldots,s_k)$ is also the lexicographically minimal shortest G-path from s_i leading to a state that satisfies $\neg\psi$. For each $s \in S$ and $B = E$ or C, let $\neg B_{G_1}\psi_1,\ldots,\neg B_{G_k}\psi_k$ be the formulas in $Sub^+(\varphi)$ such that $(M,s) \models (\neg B_{G_j}\psi_j)^{\sigma_3}$. For each state $s \in S$, we can associate a set $F(s)$ of at most n pairs $(\mathcal{H},t)$ such that $(s,t) \in \mathcal{K}_\mathcal{H}$ and for every formula $B_G\psi \in Sub(\varphi)$, if $(M,s) \models (\neg B_G\psi)^{\sigma_3}$, then there exists a pair $(\mathcal{H},t) \in F(s)$ such that t is the first state after s on the lexicographically minimal $\sigma_3(G_j)$-path from s to a state satisfying $\neg\psi$.

We can now define a set Σ of strings inductively. Let Σ_0 be the empty string. Suppose that we have constructed Σ_k consisting of strings $((s_1,t_1),i_1,\ldots,(s_k,t_k),i_k)$ with the properties given above. For each $\sigma = ((s_1,t_1),i_1,\ldots,(s_k,t_k),i_k) \in \Sigma_k$, $s \in S$, $(\mathcal{H},t) \in F(s)$, such that $\mathcal{H} \in \mathcal{A}_2$, there is exactly one string $\sigma \cdot ((s,t),i) \in \Sigma_{k+1}$. We choose $i \in A_{\mathcal{H}}$ in such a way that $i \neq i_k$, i is one of $i^0_{\mathcal{H}},\ldots,i^n_{\mathcal{H}}$, and a different i is chosen for each $(\mathcal{H},t) \in F(s)$. Since $|F(s)| \leq n$ and we can choose among $n+1$ agents $i^{\mathcal{H}}_0,\ldots,i^{\mathcal{H}}_n$, this can clearly be done. Let $\Sigma = \cup_k \Sigma_k$.

Let $M' = (S',\pi',\{\mathcal{K}_i : i \in \mathcal{A}\})$ be defined as follows:

- $S' = \cup_{\sigma\in\Sigma} S_\sigma$, where each S_σ is a disjoint copy of S. We denote by s_σ the copy of state $s \in S$ in S_σ.
- $\pi'(s_\sigma) = \pi(s)$ for $s \in S$, $\sigma \in \Sigma$.
- If $\sigma_3(i) \in \mathcal{A}_1$, define $\mathcal{K}_i = \{(s_\sigma,t_{\sigma'}) : (s,t) \in \mathcal{K}_{\sigma_3(i)}, \sigma,\sigma' \in \Sigma\}$. $\mathcal{K}_i$ is clearly reflexive, symmetric, and transitive in this case, since $\mathcal{K}_{\sigma_3(i)}$ is.
- If $\sigma_3(i) = \mathcal{H} \in \mathcal{A}_2$ and $i \in G_{\mathcal{H},j}$, then $\mathcal{K}_i = \{(s_\sigma,s_\sigma) : s \in S, \sigma \in \Sigma)\} \cup \{(s_\sigma,t_{\sigma'}),(t_{\sigma'},s_\sigma) : \sigma' = \sigma \cdot ((s,t),i^j_{\mathcal{H}})$ and $(s,t) \in \mathcal{K}_{\sigma_3(i)}\}$. Again, it is clear from the construction that $\mathcal{K}_i$ is reflexive, symmetric, and transitive.
- If $\sigma_3(i)$ is undefined, then $\mathcal{K}_i = \{(s_\sigma,s_\sigma) : s \in S, \sigma \in \Sigma)\}$. Of course, in this case $\mathcal{K}_i$ is also reflexive, symmetric, and transitive.

We claim that for each formula $\psi \in Sub^+(\varphi)$, the following are equivalent:

(a) $(M,s) \models \psi^{\sigma_3}$,

(b) $(M',s_\sigma) \models \psi$ for all $\sigma \in \Sigma$,

(c) $(M',s_\sigma) \models \psi$ for some $\sigma \in \Sigma$.

The argument proceeds by a straightforward induction on the structure of ψ. The argument that (a) implies (b) is easy using the induction hypothesis, and the implication from (b) to (c) is trivial. For the argument that (c) implies (a), the only interesting cases are when ψ is of the form $K_i\psi'$, $E_G\psi'$ or $C_G\psi'$. For $K_i\psi'$, the argument is easy because it is easy to see that $\{i\} \in \mathcal{A}_1$. For $E_G\psi'$, suppose that $(M',s_\sigma) \models E_G\psi'$. Then we must have $(M,s) \models (E_G\psi')^{\sigma_3}$. For suppose not. Then there is some $(\mathcal{H},t) \in F(s)$ such that $\mathcal{H} \in \sigma_3(G)$ and $(s,t) \in \mathcal{K}_{\mathcal{H}}$. Our construction guarantees that $\sigma' = \sigma \cdot ((s,t),i) \in \Sigma$ for some $i \in A_{\mathcal{H}}$. From Lemmas 4.3(a) and 4.4(a), it follows that $i \in G$. Moreover, by our construction, $(s_\sigma,t_{\sigma'}) \in \mathcal{K}_i$. The induction hypothesis now guarantees that $(M',t_{\sigma'}) \models \neg\psi'$. But this contradicts the assumption that $(M',s_\sigma) \models E_G\psi'$.

Finally, suppose that $(M',s_\sigma) \models C_G\psi'$. Again, for a contradiction, suppose that $(M,s) \models \neg(C_G\psi')^{\sigma_3}$. Now we proceed by a subinduction on the length of the shortest $\sigma_3(G)$-path in M leading to a state satisfying $(\neg\psi')^{\sigma_3}$ to show that $(M',s_\sigma) \models \neg C_G\psi'$. We leave the straightforward details to the reader. ∎

Next, we want an analogue of Theorem 4.9 for $\mathcal{M}^{rst}_{\mathcal{A}}$. The reader will not be surprised to learn that there are new complications here as well, although the basic result still holds.

Theorem 4.16: *If $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is finite and there is an algorithm for deciding if $i \in G$ for $G \in \mathcal{G}$ that runs in time linear in $|\mathcal{A}|$, then there is a constant $c > 0$ and an algorithm that, given a formula φ of $\mathcal{L}_{\mathcal{G}}^C$, decides if φ is satisfiable in $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rst}$ and runs in time $O(|\mathcal{A}|2^{c|\varphi|})$*

Proof: We start as in the proof of Theorem 4.9. Again, we assume for ease of exposition that $\mathcal{A}_1 \neq \emptyset$. For $i \in \mathcal{A}_1$, let S_i^1 consist of all the subsets of $ESub_i^+(\varphi)$ that are maximally consistent and let $S^1 = \cup_{i\in\mathcal{A}_1} S_i^1$. The definition of the $\mathcal{K}_i$ relations depends on whether $i \in \mathcal{A}_1$ or $i \in \mathcal{A}_2$. For $i \in \mathcal{A}_1$, we define the $\mathcal{K}_i$ relations on S^1 so that $(s,t) \in \mathcal{K}_i$ iff $s/\overline{K_i} \cup \{K_i\psi : K_i\psi \in s\} \subseteq t$ and $s/\overline{K_i} \cup \{K_i\psi : K_i\psi \in s\} = t/\overline{K_i} \cup \{K_i\psi : K_i\psi \in t\}$. It is easy to check that this modification forces these $\mathcal{K}_i$ relations to be Euclidean and transitive. For $i \in \mathcal{A}_2$ we define $\mathcal{K}_i$ so that $(s,t) \in \mathcal{K}_i$ iff $s/\overline{K_i} \subseteq t$ and $t/\overline{K_i} \subseteq s$. Clearly this modification forces these $\mathcal{K}_i$ relations to be symmetric. We force all the $\mathcal{K}_i$ relations to be reflexive just as with $\mathcal{M}_{\mathcal{A}}^r$, by eliminating $s \in S^1$ if $(s,s) \notin \mathcal{K}_i$ for some $i \in \mathcal{A}_1 \cup \mathcal{A}_2$.

We now must also change the definition of s seeming consistent. Define the relations $\preceq_i$ on $S^1 \times S_i^1$ by taking $s \preceq_i s'$ if $s' \in S_i^1$ and $s \cap ESub_i(\varphi) \subseteq s'$. Suppose that we have defined $S^1, \ldots, S^m$. S^{m+1} consists of all states $s \in S^m$ that *seem consistent*, in that the following three conditions hold (where we assume that all states considered are in S^m):

1. For all $i \in \mathcal{A}_1$, there exists an $s' \in S^j$ such that $s \preceq_i s'$.
2. There exist distinct agents $i_1, \ldots, i_k \in \mathcal{A}_1$ and states $s_1, \ldots, s_k$ such that $s \preceq_{i_h} s_h$ for $h \in \{1, \ldots, k\}$ and for every formula of the form $\neg E_G\psi \in s$, there is a t such that either
   (a) $(\exists i \in G \cap \mathcal{A}_2)((s,t) \in \mathcal{K}_i \wedge \neg\psi \in t)$ or
   (b) $(\exists h \leq k)(i_h \in G \wedge (s_h, t) \in \mathcal{K}_{i_h} \wedge \neg\psi \in t)$.
3. If $\neg C_G\psi \in s$ then there exist states $s_0, s_0', s_1, s_1', \ldots, s_k$ such that $s = s_0$, $\neg\psi \in s_k$, and there exist $j_0, \ldots, j_{k-1}$ in G such that, for each $i \leq k$, $(s_i', s_{i+1}) \in \mathcal{K}_{j_i}$ and either $j_i \in \mathcal{A}_2$ and $s_i = s_i'$ or $j_i \in \mathcal{A}_1$, $s_i \preceq_{j_i} s_i'$ and s_i' is acceptable for s_i, where we say that s' is *acceptable* for s if there are states s_h and agents i_h, $h = 1, \ldots, k$, as described in condition 2 for s, and $s' = s_i$ for some $i \leq k$.

We need to show that we can check whether s seems consistent in time $O(|\mathcal{A}|2^{|\varphi|})$. The only difficulty is to determine, for given s and s', if s' is acceptable for s. It is clear that $k \leq |\varphi|$, since we need at most one state and agent for each formula of the form $\neg E_G\psi \in s$. However, if we simply check each subgroup of states containing s' and of agents containing j where $s' \in S_j^1$ that are of size $\leq |\varphi|$ in the naive way, this check will take time at least $C(2^{|\varphi|}, |\varphi|)C(|\mathcal{A}|, |\varphi|)$ (where $C(n,k)$ is n choose k), which is unacceptable for our desired time bounds. Instead, we proceed as follows.

Suppose that $s' \in S_{i_1}^1$ and $s \preceq_{i_1} s'$. (If it is not the case that $s \preceq_{i_1} s'$, then clearly s' is not acceptable for s'.) Let $F(s,s')$ consist of all formulas $E_G\psi$ such that

1. $\neg E_G\psi \in s$,
2. $\neg\exists t, i(i \in G \cap \mathcal{A}_2 \wedge (s,t) \in \mathcal{K}_i \wedge \neg\psi \in t)$, and
3. $|A(s, s', E_G\psi)| < |\varphi|$, where $A(s, s', E_G\psi) = \{i \in G \cap \mathcal{A}_1 : i = i_1 \vee \exists t(s \preceq_i t \wedge \neg K_i\psi \in t)\}$.

Intuitively, $F(s,s')$ consists of the potentially "problematic" formulas that may prevent s' from being acceptable for s.

Let $T = \cup_{E_G\psi \in F(s)} A(s,s',E_G\psi)$. Note that $|T| < |\varphi|^2$. Suppose that $T = \{i_1,\ldots,i_N\}$. We construct sets $B_1,\ldots,B_N$ of subsets of $F(s,s')$ with the property that a set $X \in B_k$ iff there exist states $t_1,\ldots,t_k$ such that $s \preceq_{i_j} t_j$ for $j = 1,\ldots,k$, $t_1 = s'$ and, for each formula $E_G\psi \in X$, there exists a j such that $\neg K_{i_j}\psi \in t_j$.

Given a state $t \in S_i^1$, let $F_t(s,s') = \{E_G\psi \in F(s,s') : \neg K_i\psi \in t,\ i \in G\}$. Intuitively, $F_t(s,s')$ consists of the formulas in $F(s,s')$ that can be "taken care of" by state t. Let $B_1 = \{F_{s'}(s,s')\}$. Suppose that we have defined $B_1,\ldots,B_k$. Let $B_{k+1} = \{X \cup F_t(s,s') : X \in B_k \wedge s \preceq_{i_{k+1}} t\}$. It is easy to check that B_{k+1} has the required property. Moreover, we can compute the sets $B_1,\ldots,B_N$ in time $O(2^{cn})$. To see this, note that since $|F(s,s')| \leq |\varphi|$, clearly $|B_j| \leq 2^{|\varphi|}$. Thus, given B_k, we can clearly compute B_{k+1} in time $O(2^{cn})$ for some $c > 0$. Since $N < |\varphi|^2$, the result follows. Finally, we claim that s' is acceptable for s iff $F(s,s') \in B_N$.

Clearly if $F(s,s') \notin B_N$, then it is almost immediate from the definition that s' is not acceptable for s. Conversely, if $F(s) \in B_N$, then there exist states $t_1,\ldots,t_N$ such that $s' = t_1$, $s \preceq_{i_j} t_j$ and, for each formula in $E_G\psi \in F(s)$, there exists j such that $\mathcal{K}_{i_j}\psi \in t_j$. We clearly do not need all of these states and agents; we just need at most one for each formula in $F(s,s')$. That is, there exists a set $\mathcal{A}'$ of agents (contained in $\{i_1,\ldots,i_N\}$) with $|\mathcal{A}'| \leq |F(s,s')|$ and a state u_i corresponding to each agent $i \in \mathcal{A}'$ (contained in $\{t_1,\ldots,t_N\}$) such that for each formula $E_G\psi \in F(s,s')$, there exists an agent $i \in \mathcal{A}'$ such that $s \preceq_i u_i$ and $\neg K_i\psi \in u_i$. We now wish to extend $\mathcal{A}'$ to a set showing that s' is acceptable for s. If we consider any $\neg E_G\psi \in s$, either condition 2(a) is satisfied or there is already an $i \in A'$ satisfying 2(b) or $|A(s,s',E_G\psi)| \geq |\varphi|$. In the last case, it is immediate that we can extend $\mathcal{A}'$ to include an agent satisfying 2(b) for $E_G\psi$.

To show that this algorithm is correct, first suppose that φ is satisfiable. In that case, $(M,s_0) \models \varphi$ for some structure $M = (S,\pi,\{\mathcal{K}_i' : i \in \mathcal{A}\}) \in \mathcal{M}^{rst}$. As for $\mathcal{M}^{rt}$, we can associate with each state $s \in S$ and $i \in \mathcal{A}_1$ the state s_i^* in S_i^1 consisting of all the formulas $\psi \in ESub_i(\varphi)$ such that $(M,s) \models \psi$. It is easy to see that if $(s,t) \in \mathcal{K}_i'$ then $(s_i^*,t_i^*) \in \mathcal{K}_i$. Using this observation, a straightforward induction shows that the states s_i^* for $s \in S$ always seem consistent, and thus are in S^j for all j and all $i \in \mathcal{A}_1$. Moreover, $\varphi \in (s_0)_i^*$ for all $i \in \mathcal{A}_1$. Thus, the algorithm will declare that φ is satisfiable, as desired.

For the converse, we need to show that if the algorithm declares that φ is satisfiable, then it is indeed satisfiable in $\mathcal{M}^{rst}_{\mathcal{A}_1+\mathcal{A}_2}$. We need to work a little harder than in the previous proofs. Now we can no longer just view the object constructed by our algorithm as the required structure. Rather, it serves as a "blueprint" for building the required structure.

Suppose that the algorithm terminates at stage N with a state $u \in S_{i_u} = S_{i_u}^N$ containing φ. Before we go on, we make one observation that will prove useful in the sequel. Notice that if $s \preceq_i s'$, then $E_G\psi \in s$ iff $E_G\psi \in s'$ for $G \neq \{i\}$, and if $j \in \mathcal{A}_2$, then $(s,t) \in \mathcal{K}_j$ iff $(s',t) \in \mathcal{K}_j$. A *complete state* is a vector $\vec{s} = (s^i : i \in \mathcal{A}_i \wedge s^i \in S_i^N)$ such that

- $s^i \preceq_j s^j$ for all $i,j \in \mathcal{A}_1$ and
- for every formula of the form $\neg E_G\psi \in \cup_{i\in\mathcal{A}_1} s^i$, there exists an agent $j \in G$ and a state $t \in S^N$ such that $\neg\psi \in t$ and either $j \in \mathcal{A}_1 \cap G$, $\neg K_j\psi \in s^j$, and $(s^j,t) \in \mathcal{K}_j$ or $j \in \mathcal{A}_2$ and $(s^i,t) \in \mathcal{K}_j$ for some $i \in \mathcal{A}_1$ (and hence $(s^i,t) \in \mathcal{K}_j$ for all $i \in \mathcal{A}_1$).

By consistency condition 2, every state $s \in S^N$ must be a component of some (perhaps many) complete states.

Define a structure $M^* = (S^*, \pi^*, \{\mathcal{K}_i^* : i \in \mathcal{A}_1 \cup \mathcal{A}_2\}$ as follows:

- S^* consists of all complete states;
- $\pi^*(\vec{s})(p) = \mathbf{true}$ iff $p \in \cup_{i \in \mathcal{A}_1} s^i$;
- $(\vec{s}, \vec{t}) \in \mathcal{K}_i^*$ for $i \in \mathcal{A}_1$ iff $s^i = t^i$ or $(s^i, t^i) \in \mathcal{K}_i$;
- $(\vec{s}, \vec{t}) \in \mathcal{K}_i^*$ for $i \in \mathcal{A}_2$ iff $(s^j, t^j) \in \mathcal{K}_i$ for some $j \in \mathcal{A}_1$ (it is easy to check that if $(s^j, t^j) \in \mathcal{K}_i$ for some $j \in \mathcal{A}_1$ then $(s^j, t^j) \in \mathcal{K}_j$ for all $j \in \mathcal{A}_1$).

It is easy to check that $M^* \in \mathcal{M}^{rst}_{\mathcal{A}_1 + \mathcal{A}_2}$. We now show that for all $\psi \in \cup_{i \in \mathcal{A}_i} ESub_i^+(\varphi)$, we have

$$(M^*, \vec{s}) \models \psi \text{ iff } \psi \in \cup_{i \in \mathcal{A}_1} s^i.$$

We proceed, as usual, by induction on the structure of ψ. If ψ is a primitive proposition, a conjunction, or a negation, the argument is easy. Suppose that ψ is of the form $E_G\psi'$. If $E_G\psi' \in \cup_{i \in \mathcal{A}_1} s^i$, then the construction of the $\mathcal{K}_j$ relations guarantees that $\psi' \in \cup_{i \in \mathcal{A}_1} t^i$ for all $\vec{t} \in S^*$ such that $(\vec{s}, \vec{t}) \in \mathcal{K}_j^*$ for some $j \in G$. Thus, by the induction hypothesis, we have that $(M^*, \vec{s}) \models E_G\psi'$. For the converse, suppose that $\neg E_G\psi' \in \cup_{i \in \mathcal{A}_1} s^i$. Then from the definition of complete state and consistency condition 2, there must be some complete state $\vec{t}$ and $j \in G$ such that $(\vec{s}, \vec{t}) \in \mathcal{K}_j$ and $\neg\psi' \in \cup_{i \in \mathcal{A}_1} t^i$.

Finally, suppose that ψ is of the form $C_G\psi'$. If $C_G\psi' \in \cup_{i \in \mathcal{A}_1} s^i$ then, since $E_G(\psi' \wedge C_G\psi')$ must also be in $\cup_{i \in \mathcal{A}_1} s^i$, an easy induction on the length of the path shows that for every complete state $\vec{t}$ G-reachable from $\vec{s}$, we must have $\psi' \in \cup_{i \in \mathcal{A}_1} t^i$ so, by the induction hypothesis, we have $(M^*, \vec{s}) \models C_G\psi'$. For the converse, suppose that $\neg C_G\psi \in \cup_{i \in \mathcal{A}_1} s^i$. Then $\neg C_G\psi \in s^j$ for some (in fact, all) $j \in \mathcal{A}_1$. If $G \cap \mathcal{A}_1 \neq \emptyset$, choose $j \in G \cap \mathcal{A}_1$; otherwise, choose $j \in G$ arbitrarily. From consistency condition 3, it easily follows that there exist complete states $\vec{s}_0, \ldots, \vec{s}_k$ and $j_0, \ldots, j_{k-1} \in G$ such that $s_0^j = s^j$, $(\vec{s}_h, \vec{s}_{h+1}) \in \mathcal{K}^*_{j_h}$ $h = 0, \ldots, k-1$, and $\neg\psi' \in \cup_{i \in \mathcal{A}_1} s_k^i$. If $j \in \mathcal{A}_1$, then $(\vec{s}, \vec{s}_0) \in \mathcal{K}_j^*$; if $j \notin \mathcal{A}_1$, then $j_0 \in \mathcal{A}_2$, and it follows from our initial observation that $(\vec{s}, \vec{s}_1) \in \mathcal{K}^*_{j_0}$. In either case, $\vec{s}_k$ is G-reachable from $\vec{s}$, so $(M^*, \vec{s}) \models \neg C_G\psi'$, as desired. ∎

We can now prove Theorem 3.4 for $\mathcal{M}^{rst}_{\mathcal{A}}$.

Proof of Theorem 3.4 for $\mathcal{M}^{rst}_{\mathcal{A}}$: Again, the lower bound follows from standard results in [HM92].

For the upper bound, suppose that we are given a formula φ such that $n = |\varphi|$. We can compute the set $\mathcal{E}_n^m(\mathcal{G}_\varphi)$ defined just before Lemma 4.11 in time $O(n^2 2^{cn})$, using at most $n^2 2^n$ calls to the oracle O_m, just as we computed $\mathcal{E}^1(\mathcal{G}_\varphi)$. Similarly, we can characterize the sets $\mathcal{H}$ such that $(G, \mathcal{H})$ is in $\mathcal{R}(\mathcal{G}_\varphi^m) = \mathcal{R}^m(\mathcal{G}_\varphi)$ by a pair $(\mathcal{H}', X)$, where $\mathcal{H}' \subseteq \mathcal{G}_\varphi$ and X is either $\emptyset$ or an element of $\mathcal{E}_m^1(\mathcal{G}_\varphi)$ and compute which of the pairs actually represent sets in $\mathcal{H}$ such that $(G, \mathcal{H}) \in \mathcal{R}(\mathcal{G}_\varphi^m)$ using at most $2n(2^{2n} + 2^n)$ calls to the oracle O_m. We cannot compute the individual elements of the sets $A_{\mathcal{H}}$ such that $|A_{\mathcal{H}}| \leq m$, but it does not matter. It suffices that we know the cardinality of these atoms (which our oracle will tell us). We let $\mathcal{A}_1$ consist of the agents $i_1^{\mathcal{H}}, \ldots, i_{|A_{\mathcal{H}}|}^{\mathcal{H}}$ for each $\mathcal{H}$ such that $|A_{\mathcal{H}}| \leq |\varphi|$ ($i_1^{\mathcal{H}}, \ldots, i_{|A_{\mathcal{H}}|}^{\mathcal{H}}$ are just fresh names for agents); let $\mathcal{A}_2$ consist of all $\mathcal{H}$ such that $|A_{\mathcal{H}}| > |\varphi|$.

It is now straightforward to compute the formula φ^{σ_3} in time $O(2^{cn})$ using $O(2^{cn})$ oracle calls. We now apply Proposition 4.15 and Theorem 4.16, just as we applied Proposition 4.1 and Theorem 4.5 in the case of $\mathcal{M}_{\mathcal{A}}$. ∎

We now turn our attention to proving Theorem 3.1 for $\mathcal{M}_{\mathcal{A}}^{rst}$. Again, the basic structure is the same as for $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}^{rt}$.

Lemma 4.17: *The mapping σ_3 (when viewed as a map with domain $2^{\mathcal{A}}$) is injective on $\widetilde{\mathcal{G}}_{\varphi}^{n}$.*

Let $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2}$ consist of the axioms in $(\mathrm{T}_{\mathcal{G}}^{C})^{+}$ (including E5) together with E6 and every instance of K4 and K5 for $i \in \mathcal{A}_1$. We write $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \psi$ if there is a proof of ψ in $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2}$ using only the modal operators that appear in φ and K_i for $i \in \mathcal{A}_1$.

Lemma 4.18: *If $\mathcal{A}$ is finite and $\varphi \in \mathcal{L}_{\mathcal{G}}^{C}$ is valid with respect to $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{rst}$, then $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi$.*

Proof: The proof is similar in spirit to that of Lemma 4.13 for $\mathcal{M}_{\mathcal{A}}^{rt}$, except that since we have a different definition of the $\mathcal{K}_i$ relations and of seeming consistent, we must check that states eliminated under this definition are inconsistent. Again we must consider each of the three ways that a state s can be eliminated.

First, suppose that $s \in S^j$ and, for some $i \in \mathcal{A}_1$, there is no s' such that $s \preceq_i s'$. As before, propositional reasoning shows that $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_s \Leftrightarrow \vee_{\{s' \in S_i^1 : s \preceq_i s'\}} \varphi_{s'}$. Thus, it easily follows that $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \neg\varphi_s$.

Next, suppose that $s \in S^j$ does not satisfy the second condition of seeming consistent. There must be a formula $\neg E_G\psi \in s$ such that

1. for all $i \in \mathcal{A}_2$ and all $t \in S^j$ such that $(s,t) \in \mathcal{K}_i$, we have $\psi \in t$ and
2. for all $i \in G \cap \mathcal{A}_1$, and all $t \in S^j$ such that $(s^i, t) \in \mathcal{K}_i$, we have $\psi \in t$.

Define an *extension of* s to be a vector $\vec{s} = (s^i : i \in \mathcal{A}_1)$ of states, where $s \preceq_i s^i$. Let $EX(s)$ be the set of all extensions of s. If $\vec{s}$ is an extension of s, let $\varphi_{\vec{s}}$ be the conjunction over all $i \in \mathcal{A}_1$ of the formulas in φ_{s^i}. By straightforward propositional reasoning, we have $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_s \Leftrightarrow \vee_{\vec{s} \in EX(s)} \varphi_{\vec{s}}$. Thus, to show that $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \neg\varphi_s$ if s is eliminated by the second condition of seeming consistent, it suffices to show that $(\mathrm{S5}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \neg\varphi_{\vec{s}}$ for each $\vec{s} \in EX(s)$. This we do by showing that $(\mathrm{S5}_{G}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_{\vec{s}} \Rightarrow E_G\psi$ for each $\vec{s} \in EX(s)$.

So suppose that $\vec{s} \in EX(s)$. The proof follows the lines of the analogous argument in the proof of Lemma 4.13. As before, it suffices to find, for each $i \in G$ and each $t \in S_i^j$ with $\neg\psi \in t$, a set $G^{i,t}$ of agents containing i such that $(\mathrm{S5}_{G}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_{\vec{s}} \Rightarrow E_{G^{i,t}}\neg\varphi_t$. For $i \in \mathcal{A}_2$, this follows as before if the reason that $(s,t) \notin \mathcal{K}_i$ is that $s/\overline{K_i} \not\subseteq t$. If instead $t/\overline{K_i} \not\subseteq s$, then there is some $E_{G'}\theta \in t$ with $i \in G'$ such that $\neg\theta \in s$ and so $\neg\theta \in s^i$ for each i. Thus $(\mathrm{S5}_{G}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_{\vec{s}} \Rightarrow \neg\theta$ and, by E6, $(\mathrm{S5}_{G}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \neg\theta \Rightarrow E_{G'}\neg E_{G'}\theta$. Since $E_{G'}\theta \in t$ we have that $(\mathrm{S5}_{G}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_{\vec{s}} \Rightarrow E_{G'}\neg\varphi_t$. That is, we can take $G^{i,t} = G'$ in this case.

For $i \in \mathcal{A}_1$, we show that $(\mathrm{S5}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_{\vec{s}} \Rightarrow K_i \neg\varphi_t$ (so that we can take $G^{i,t} = \{i\}$). By our assumption, $(s^i, t) \notin \mathcal{K}_i$ for all $t \in S^i_j$. Thus, if $t \in S^i_j$, there is some formula θ such that either $K_i\theta \in s^i$ and $\neg K_i\theta \in t$ or $K_i\theta \in t$ and $\neg K_i\theta \in s^i$. Here we are implicitly using the following facts: (1) if $E_{G'}\theta \in s$ for some G' such that $i \in G'$ then $K_i\theta \in s^i$, since $s^i \in S^1_i$, and similarly for t, (2) if $K_i\theta \notin s$, then $\neg K_i\theta \in s$, since $s^i \in S^j_i$, and similarly for t, and (3) if $K_i\theta \in s^i$ then $\theta \in S$ since $(s, s) \in \mathcal{K}_i$, and similarly for t. If $K_i\theta \in s$ and $\neg K_i\theta \in t$, it follows that $(\mathrm{S5}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\neg\varphi_t$ just as in the case of $(\mathrm{S4}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2}$. If $K_i\theta \in t$ and $\neg K_i\theta \in s$, then by K5 we have $(\mathrm{S5}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow K_i\neg K_i\theta$ $(\mathrm{S5}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg K_i\theta \Rightarrow \neg\varphi_t$. The desired result now follows by standard arguments.

We have now shown that for all $i \in G$ and $t \in S^i_j$ such that $\psi \in t$, there exists some set $G^{i,t}$ with $i \in G^{i,t}$ such that $(\mathrm{S5}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow E_{G^{i,t}}\neg\varphi_t$. We can now conclude that $(\mathrm{S5}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow \neg E_G\psi$ just as in the case of $(\mathrm{S4}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2}$, showing that φ_s is inconsistent, as desired.

Finally, if $s \in S^j$ does not satisfy the third condition of seeming consistent, the argument that $(\mathrm{S5}^C_{\mathcal{G}})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg\varphi_s$ is similar to that of Lemma 4.7. We replace G-reachability by the existence of sequences as in condition 3 in the definition of seeming consistent in Theorem 4.16 and note that we have essentially already proved the analogue of (6) from Lemma 4.7. We leave the remaining details to the reader. ∎

Proof of Theorem 3.1 for $\mathcal{M}^{rst}_{\mathcal{A}}$: The proof follows as for $\mathcal{M}^{rt}_{\mathcal{A}}$ using the analogous lemmas proved above for $\mathcal{M}^{rst}_{\mathcal{A}}$. ∎

4.4 Dealing with $\mathcal{M}^{elt}_{\mathcal{A}}$

For $\mathcal{M}^{elt}_{\mathcal{A}}$, we proceed much as for $\mathcal{M}^{rst}_{\mathcal{A}}$. There is one new subtlety. Consider the construction in the proof of Proposition 4.15, which uses σ_3. Recall that $\sigma_3(i)$ may be undefined for some i. For such i, we defined $\mathcal{K}_i$ to consist of all pairs (s_σ, s_σ), making it reflexive. This approach will not work for $\mathcal{M}^{elt}_{\mathcal{A}}$. More precisely, the analogue of Proposition 4.15 for $\mathcal{M}^{elt}_{\mathcal{A}}$ will not hold using this construction (even if we drop the reflexivity requirement). For example, if $\varphi = \neg p \wedge E_{G_1}p \wedge E_{G_2}p$ and $G_1 \cap G_2 \neq \emptyset$, then φ^{σ_3} is satisfiable in $\mathcal{M}^{elt}_{\mathcal{A}_1+\mathcal{A}_2}$ but φ is not satisfied in the structure M' constructed in Proposition4.15, since for all $i \in G_1 \cap G_2$, the construction will make $\mathcal{K}_i$ reflexive. We solve this problem by defining a mapping σ_4 much like σ_3, except that we ensure that σ_4 is never undefined.

Let $\mathcal{B}$ be the set of maximal subsets $\mathcal{T}$ of $\mathcal{G}_\varphi$ such that $\cap\mathcal{T} \neq \emptyset$ and such that the corresponding atom over $\mathcal{G}^\varphi$, $\mathcal{A}_{\mathcal{T}} = (\cap\mathcal{T}) \cap (\cap_{G \in \mathcal{G}_\varphi - \mathcal{T}} G)$ ($= \cap\mathcal{T}$ by the maximality of $\mathcal{T}$), is not one of the ones $\mathcal{A}_{\mathcal{H}}$ for $\mathcal{H} \in \mathcal{A}^{\varphi,rst}$. Let $\mathcal{A}^{\varphi,elt} = \mathcal{A}^{\varphi,rst} \cup \mathcal{B}$, $\mathcal{A}_1 = \mathcal{B} \cup \{\mathcal{H} \in \mathcal{A}^{\varphi,rst} : |\mathcal{A}_{\mathcal{H}}| = 1\}$, $\mathcal{A}_2 = \mathcal{A}^{\varphi,elt} - \mathcal{A}_1$. The definitions of $\sigma_4 : \mathcal{A} \to \mathcal{A}^{\varphi,elt}$ and $\tau_4 : \mathcal{A}^{\varphi,elt} \to 2^{\mathcal{A}}$ need some care. If $i \in \mathcal{A}_{\mathcal{H}}$ for some $\mathcal{H} \in \mathcal{A}^{\varphi,rst}$, let $\sigma_4(i) = \mathcal{H}$ as before. Otherwise, choose $\mathcal{T} \in \mathcal{B}$ such that $\mathcal{T} \supseteq \{G \in \mathcal{G}_\varphi : i \in G\}$ and let $\sigma(i) = \mathcal{T}$. Note that, by construction, σ_4 is defined for all i. For $\mathcal{H} \in \mathcal{A}^{\varphi,rst}$, $\tau_4(\mathcal{H}) = \cap\{\widetilde{\mathcal{G}}^n_\varphi - \mathcal{H}\}$ as before. For $\mathcal{T} \in \mathcal{B}$, choose some $i_{\mathcal{T}} \in \mathcal{A}_{\mathcal{T}}$ (it does not matter which) and set $\tau_4(\mathcal{T}) = \{i_{\mathcal{T}}\}$.

Proposition 4.19: *φ is satisfiable in $\mathcal{M}^{elt}_{\mathcal{A}}$ iff φ^{σ_4} is satisfiable in $\mathcal{M}^{elt}_{\mathcal{A}_1+\mathcal{A}_2}$.*

Proof: First suppose that $(M,s) \models \varphi$, where $M \in \mathcal{M}^{elt}$. We convert M to $M' \in \mathcal{M}^{elt}_{\mathcal{A}_1+\mathcal{A}_2}$ as before by defining $\mathcal{K}_{\mathcal{I}} = \cup\{\mathcal{K}_i : i \in \tau_4(\mathcal{I})\}$ for $\mathcal{I} \in \mathcal{A}^{\varphi,elt}$. To apply Proposition 4.1, we need to show that $\cup\{\tau(\mathcal{I}) : \mathcal{I} \in \sigma_4(G)\} = G$ for all $G \in \mathcal{G}_\varphi$. We know from the analysis of the $\mathcal{M}^{rst}$ case that $\cup\{\tau_3(\mathcal{H}) : \mathcal{H} \in \sigma_3(G)\} = G$ for all $G \in \mathcal{G}_\varphi$. Since $\sigma_4(G) \supseteq \sigma_3(G)$ and $\tau_4(\mathcal{H}) = \tau_3(\mathcal{H})$ for $\mathcal{H} \in \mathcal{A}^{\varphi,rst}$, we have that $\cup\{\tau_4(\mathcal{I}) : \mathcal{I} \in \sigma_4(G)\} = \cup\{\tau_3(\mathcal{H}) : \mathcal{H} \in \sigma_3(G) \cup \cup\{\tau_3(\mathcal{I}) : \mathcal{I} \in \sigma_4(G) - \sigma_3(G)\}$. It is clear from the definitions, however, that if $\mathcal{I} \in \sigma_4(G) - \sigma_3(G)$, then there exists some $i \in G$ such that $\mathcal{I} = \sigma_4(i)$ and $\sigma_3(i)$ is undefined. Moreover, $\mathcal{I} = \mathcal{A}_{\mathcal{T}} \subseteq G$, so $\tau_4(\mathcal{I}) \in G$. Thus, $\cup\{\tau_3(\mathcal{I}) : \mathcal{I} \in \sigma_4(G) - \sigma_3(G)\} \subseteq G$, so $\cup\{\tau_4(\mathcal{I}) : \mathcal{I} \in \sigma_4(G)\} = \cup\{\tau_3(\mathcal{H}) : \mathcal{H} \in \sigma_3(G)\} = G$, as desired. Applying Proposition 4.1, we get that to see that $(M',s) \models \varphi^{\sigma_4}$.

It remains to verify that $M' \in \mathcal{M}^{elt}_{\mathcal{A}_1+\mathcal{A}_2}$. For this, we need to show that the $\mathcal{K}_{\mathcal{I}}$ relations for $\mathcal{I} \in \mathcal{A}_1$ are Euclidean, serial and transitive and that those in $\mathcal{A}_2$ are serial and secondarily reflexive. For the ones in $\mathcal{A}_1$, note that $\tau_4(\mathcal{I})$ is a singleton and so the desired properties hold since they hold for all agents in M. For the ones in $\mathcal{A}_2$, we just note that the union of serial relations is serial and the union of Euclidean relations is secondarily reflexive.

For the other direction, we proceed much as in the proof of Proposition 4.15. In addition to the concerns dealt with there for $\mathcal{M}^{rst}$, our primary new one is to make sure that the $\mathcal{K}_i$ relations for all agents are serial. The problem arises for those i for which $\sigma_3(i)$ was undefined. The new agents in $\mathcal{B}$ are used to deal with this problem.

We proceed much as in Proposition 4.15, with two changes. First, we replace the automatic forcing of reflexivity by forcing secondary reflexivity for $\sigma_3(i) \in \mathcal{A}_2$. Second, we modify the definition of the $\mathcal{K}_i$ relation in M' as follows.

- If $\sigma_4(i) \in \mathcal{A}_1 \cap \mathcal{A}^{\varphi,rst}$ then, as before, $\mathcal{K}_i = \{(s_\sigma, t_{\sigma'}) : (s,t) \in \mathcal{K}_{\sigma_3(i)}, \sigma, \sigma' \in \Sigma\}$.
- If $\sigma_4(i) \in \mathcal{A}_2$ and $i \in G_{\mathcal{H},j}$, then $\mathcal{K}_i = \{(s_\sigma, t_{\sigma'}), (t_{\sigma'}, t_{\sigma'}) : \sigma' = \sigma \cdot ((s,t), i^j_{\mathcal{H}})\}$.
- If $\sigma_4(i) = \mathcal{T} \in \mathcal{B}$, then $\mathcal{K}_i = \{(s_\sigma, t_{\sigma'}) : (s,t) \in \mathcal{K}_{\sigma_4(i)}, \sigma, \sigma' \in \Sigma\}$.

Now note that every relation $\mathcal{K}_i$ is Euclidean, serial and transitive. For the ones corresponding to agents in $\mathcal{A}_1$ this is immediate from the fact that the agents in $\mathcal{A}_1$ have these properties. For those with $\sigma_4(i) \in \mathcal{A}_2$, seriality follows from the fact that the agents in $\mathcal{A}_2$ are serial and the construction. Transitivity and the Euclidean property follow from the construction. In particular, if there is a $\mathcal{K}_i$ edge coming into some t_σ then there is none going out by construction except for the one from t_σ to itself.

The verification that M' satisfies φ now proceeds as in Proposition 4.15. ∎

Theorem 4.20: *If $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}$ is finite and there is an an algorithm for deciding if $i \in G$ for $G \in \mathcal{G}$ that runs in time linear in $|\mathcal{A}|$, then there is a constant $c > 0$ (independent of $|\mathcal{A}|$) and an algorithm that, given a formula φ of $\mathcal{L}^C_{\mathcal{G}}$, decides if φ is satisfiable in $\mathcal{M}^{elt}_{\mathcal{A}_1+\mathcal{A}_2}$ and runs in time $O(|\mathcal{A}|2^{c|\varphi|})$.*

Proof: The argument here is like that for the $\mathcal{M}^{rst}_{\mathcal{A}_1+\mathcal{A}_2}$ case in Theorem 4.16. We keep the definition of $\mathcal{K}_i$ for $i \in \mathcal{A}_1$ and, as we noted there, this makes these relations Euclidean and transitive. We change the definition of $\mathcal{K}_i$ for $i \in \mathcal{A}_2$ by putting (s,t) in $\mathcal{K}_i$ iff $s/\overline{K_i} \subseteq t$ and $t/\overline{K_i} \subseteq t$. This latter definition clearly makes the $\mathcal{K}_i$ secondarily reflexive for $i \in \mathcal{A}_2$. We ensure seriality by adding a clause to the definition of a state s seeming consistent:

4 For every agent $i \in \mathcal{A}_2$ there is a state t such that $(s,t) \in \mathcal{K}_i$ and for every agent $i \in \mathcal{A}_1$ there are states s' and t such $s \preceq_i s'$ and $(s',t) \in \mathcal{K}_i$.

The proof now proceeds as before. ∎

Proof of Theorem 3.4 for $\mathcal{M}_{\mathcal{A}}^{elt}$: The argument here is essentially the same as for $\mathcal{M}_{\mathcal{A}}^{rst}$. Just note that using the oracle O' we can determine the members of $\mathcal{B}$ within the appropriate time bound and so compute φ^{σ_4} as required. ∎

We now turn our attention to proving Theorem 3.1 for $\mathcal{M}_{\mathcal{A}}^{elt}$. The basic structure is the same as for $\mathcal{M}_{\mathcal{A}}^{rst}$.

Lemma 4.21: *The mapping σ_4 (when viewed as a map with domain $2^{\mathcal{A}}$) is injective on $\widetilde{\mathcal{G}}_{\varphi}^{n}$.*

Let $(\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2}$ consist of the axioms in $\mathrm{K}_{\mathcal{G}}^{C}$ together with K3, E4, E7, and every instance of K4 and K5 for $i \in \mathcal{A}_1$. We write $(\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \psi$ if there is a proof of ψ in $(\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2}$ using only the modal operators that appear in φ and K_i for $i \in \mathcal{A}_1$.

Lemma 4.22: *If $\mathcal{A}$ is finite and $\varphi \in \mathcal{L}_{\mathcal{G}}^{C}$ is valid with respect to $\mathcal{M}_{\mathcal{A}_1+\mathcal{A}_2}^{elt}$, then $(\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi$.*

Proof: The proof is similar to that of Lemma 4.18 for $\mathcal{M}_{\mathcal{A}}^{rst}$. Again we must check that all states eliminated in the construction are provably inconsistent, but now using the axioms of $(\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2}$ and the modified definition of the $\mathcal{K}_i$ relations, and dealing with the additional clause in the definition of seeming consistent.

The argument for the first condition for seeming consistent is the same as that for $\mathcal{M}_{\mathcal{A}}^{rst}$.

Before dealing with the second condition, we prove a fact that will also be useful in dealing with the fourth condition. Let $T_i = \{t \in S_i^j : (t,t) \in \mathcal{K}_i\}$. It is easy to see that

$$\text{if } t \notin T_i\text{, then } (\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_s \Rightarrow E_G \neg\varphi_t \text{ for some } G \text{ such that } i \in G. \tag{13}$$

For if $t \notin T_i$, then there exists $E_G\theta \in t$ such that $i \in G$ and $\neg\theta \in t$. But then $(E_G\theta \Rightarrow \theta) \Rightarrow \neg\varphi_t$ is propositionally valid (and so provable by Prop). Since $(\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_s \Rightarrow E_G(E_G\theta \Rightarrow \theta)$, we can easily obtain (13) using (4).

Now suppose that s is eliminated because it does not satisfy the second condition for seeming consistent due to $E_G\psi$. It again suffices to show that for each $i \in G$ and $t \in S_i^j$ such that $\psi \in t$, there is a set $G^{i,t}$ of agents containing i such that $(\mathrm{KD45}_{\mathcal{G}}^{C})^{\mathcal{A}_1+\mathcal{A}_2} \vdash_{\varphi} \varphi_{\vec{s}} \Rightarrow E_{G^{i,t}}\neg\varphi_t$. First suppose $i \in \mathcal{A}_2$. If $(s,t) \notin \mathcal{K}_i$ because $s/\overline{K_i} \not\subseteq t$ then the argument given in Lemma 4.7 works to get a $G^{i,t}$ as desired. If $s/\overline{K_i} \subseteq t$ but $t/\overline{K_i} \not\subseteq t$ then the existence of the required $G^{i,t}$ is immediate from (13). Now suppose $i \in \mathcal{A}_1$ and t is such that $(s^i,t) \notin \mathcal{K}_i$. If $s/K_i \not\subseteq t$, then there is some formula θ such that $K_i\theta \in s^i$ and $\neg\theta \in t$; it easily follows that $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_{\vec{s}} \Rightarrow K_i\neg\varphi_t$, as required. If $\{K_i\theta : K_i\theta \in s\} \not\subseteq t$, then there is some θ such that $K_i\theta \in s$ but $\neg K_i\theta \in t$; the result now easily follows using K4, just as in the argument for $(\mathrm{S4}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2}$. If both of these conditions hold (but still $(s^i, t) \notin \mathcal{K}_i$), then it must be that there is a θ with $K_i\theta \in t$ and $K_i\theta \notin s$. In this case $\neg K_i\theta \in s$, and the result follows using K5, just as in the argument for $(\mathrm{S5}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2}$.

The argument in the case that s is eliminated because it does not satisfy the third condition for seeming consistent is the same as in the proof of Lemma 4.18.

Finally, suppose that s does not satisfy the new (fourth) condition of seeming consistent. Then either

- there is an $i \in \mathcal{A}_2$ for which there is no t with $(s,t) \in \mathcal{K}_i$ or
- there is an $i \in \mathcal{A}_1$ for which there is no pair s', t such that $s \preceq_i s'$ and $(s', t) \in \mathcal{K}_i$.

For the first case, for each $t \in T_i$, it must be the case that $s/\overline{K_i} \not\subseteq t$, so that there must be some $G^{i,t}$ with $i \in G^{i,t}$ such that $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash \varphi_s \Rightarrow E_{G^{i,t}}\neg\varphi_t$, as usual. By (13), for each $t \notin T_i$, there is some $G^{i,t}$ with $i \in G^{i,t}$ such that $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow E_{G^{i,t}}\neg\varphi_t$. Thus, $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_s \Rightarrow \wedge_{t \in S^j} E_{G^{i,t}}\neg\varphi_t$. But since $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg(\wedge_{t\in S^j}\neg\varphi_t)$ by induction and propositional reasoning, it follows from E7 that $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg(\wedge_{t\in S^j} E_{G^{i,t}}\neg\varphi_t)$. Thus we get $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg\varphi_s$, as desired.

For the second case, we know as in the proof of Lemma 4.18 that φ_s is provably equivalent to the disjunction of $\varphi_{s'}$ for those s' such that $s \preceq_i s'$ and similarly for any t. Thus to prove $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg\varphi_s$ it suffices to prove $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg\varphi_{s'}$ for every $s' \in S_i^j$ such that $s \preceq_i s'$. For each such s' we know that there is no $t' \in S_i^j$ such that $(s', t') \in \mathcal{K}_i$. Given s', if $t' \in S_i^j$ and $(s', t') \notin \mathcal{K}_i$, then the same argument as in the proof of Lemma 4.18 shows that $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_{s'} \Rightarrow K_i\neg\varphi_{t'}$, since the $\mathcal{K}_i$ relations are defined the same way for agents in $\mathcal{A}_1$ in both the $\mathcal{M}_{\mathcal{A}}^{elt}$ and $\mathcal{M}_{\mathcal{A}}^{rst}$ cases, and the proof in Lemma 4.18 used only axioms K4 and K5 (as well as Prop, K1, and MP), and these axioms are in both $(\mathrm{S5}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2}$ and $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2}$.

By (3), we have that $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash \varphi_{s'} \Rightarrow K_i(\wedge_{t'\in S_i^j}\neg\varphi_{t'})$. Since $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi (\wedge_{t'\in S_i^j}\neg\varphi_{t'}) \Rightarrow \mathit{false}$ by induction and propositional reasoning, we conclude that $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \varphi_{s'} \Rightarrow K_i\mathit{false}$. Now using K3, we get $(\mathrm{KD45}_{\mathcal{G}}^C)^{\mathcal{A}_1+\mathcal{A}_2} \vdash_\varphi \neg\varphi_{s'}$, as desired. ∎

Proof of Theorem 3.1 for $\mathcal{M}_{\mathcal{A}}^{elt}$: The proof follows as for $\mathcal{M}_{\mathcal{A}}^{rt}$ using the analogous lemmas proved above for $\mathcal{M}_{\mathcal{A}}^{elt}$. We must just show that E7 is derivable from the other axioms in $\mathrm{KD45}_{\mathcal{G}}^C$. Suppose that $i \in G_1 \cap \ldots G_k$. Then, using E1, $\mathrm{KD45}_{\mathcal{G}}^C \vdash E_{G_1}\varphi_1 \wedge \ldots \wedge E_{G_k}\varphi_k \Rightarrow K_i\varphi_1 \wedge \ldots \wedge K_i\varphi_k$. By (3), we have $\mathrm{KD45}_{\mathcal{G}}^C \vdash K_i\varphi_1 \wedge \ldots \wedge K_i\varphi_k \Rightarrow K_i(\varphi_1 \wedge \ldots \wedge \varphi_k)$. Thus, $\mathrm{KD45}_{\mathcal{G}}^C \vdash \neg K_i(\varphi_1 \wedge \ldots \wedge \varphi_k) \Rightarrow \neg(E_{G_1}\varphi_1 \wedge \ldots \wedge E_{G_k}\varphi_k)$. It thus suffices to show that in $\mathrm{KD45}_{\mathcal{G}}^C$, from $\neg(\varphi_1 \wedge \ldots \wedge \varphi_k)$ we can infer $\neg K_i(\varphi_\wedge \ldots \wedge \varphi_k)$. But since $\neg(\varphi_1 \wedge \ldots \wedge \varphi_k)$ is equivalent to $(\varphi_1 \wedge \ldots \varphi_k) \Rightarrow \mathit{false}$, this follows easily using (4) and K3. ∎

4.5 The complexity of querying the oracles

Up to now we have assumed that we are charged one for each query to an oracle. In this section, we reconsider our results, trying to take into account more explicitly the cost of the oracle queries.

Let $f(m,k)$ be the worst-case time complexity of deciding whether a set with description $G \in \widehat{\mathcal{G}}^m_{\mathcal{A}}$ such that $l(G) \leq k$ has cardinality greater $m' \leq m$ (where we take the worst case over all $G \in \widehat{\mathcal{G}}^m_{\mathcal{A}}$ such that $l(G) \leq k$ and over all $m' \leq m$). Let $g(k)$ to be the worst-case complexity of deciding if $G_1 \cap \ldots \cap G_k = \emptyset$ for $G_1, \ldots, G_k \in \mathcal{G}_{\mathcal{A}}$. We take $f(m,k)$ (resp., $g(k)$) to be ∞ if these questions are undecidable. We can think of $f(m,k)$ (resp., $g(k)$) as the worst-case cost of querying the oracle O_m (resp., O') on a set with a description of length $\leq k$.

Using these definitions, we can sharpen Theorem 3.4 as follows.

Theorem 4.23: *There is a constant $c > 0$ and an algorithm that decides if a formula $\varphi \in \mathcal{L}^C_{\mathcal{G}}$ is satisfiable in $\mathcal{M}_{\mathcal{A}}$ (resp., $\mathcal{M}^r_{\mathcal{A}}$, $\mathcal{M}^{rt}_{\mathcal{A}}$, $\mathcal{M}^{rst}_{\mathcal{A}}$, $\mathcal{M}^{elt}_{\mathcal{A}}$) and runs in time $2^{c|\varphi|}f(0,|\varphi|)$ (resp., $2^{c|\varphi|}f(0,|\varphi|)$, $2^{c|\varphi|}f(1,2^{c|\varphi|^2})$, $2^{c|\varphi|}f(|\varphi|,2^{c|\varphi|^2})$, $2^{c|\varphi|}(f(|\varphi|,2^{c|\varphi|^2})+g(|\varphi|))$) Moreover, if $\mathcal{G}$ contains a subset with at least two elements, then there exists a constant $d > 0$ such that every algorithm for deciding the satisfiability of formulas in $\mathcal{M}_{\mathcal{A}}$ (resp., $\mathcal{M}^r_{\mathcal{A}}$, $\mathcal{M}^{rt}_{\mathcal{A}}$, $\mathcal{M}^{rst}_{\mathcal{A}}$, $\mathcal{M}^{elt}_{\mathcal{A}}$) runs in time at least* $\max(2^{d|\varphi|}, f(0,d|\varphi|))$ *(resp., (*$\max(2^{d|\varphi|}, f(0,d|\varphi|))$, $\max(2^{d|\varphi|}, f(1,d|\varphi|))$, $\max(2^{d|\varphi|}, f(d|\varphi|,d|\varphi|))$, $\max(2^{d|\varphi|}, f(d|\varphi|,d|\varphi|), g(d|\varphi|))$*) for infinitely many formulas φ.*

Proof: The upper bound is almost immediate from the proof of Theorem 3.4. The only point that needs discussion is the second argument—$2^{c|\varphi|^2}$—of f in the cases $\mathcal{M}^{rt}_{\mathcal{A}}$, $\mathcal{M}^{rst}_{\mathcal{A}}$, and $\mathcal{M}^{elt}_{\mathcal{A}}$. This follows from Lemma 4.11. An easy induction on i shows that the sets in the set $\mathcal{E}^{|\varphi|}_i$ constructed just before Lemma 4.11 have description length at most $\leq 2^{2i|\varphi|}$ (using the fact that $|\mathcal{E}^{|\varphi|}_i| \leq 2^{|\varphi|}$ for all i). Thus, all the sets that we need to deal with have description length $\leq 2^{2|\varphi|^2}$, since they are all in $\mathcal{E}^{|\varphi|}_{|\varphi|}$, by Lemma 4.11(e).

The lower bound is immediate from the results of [HM92] and Proposition 3.3. ∎

Note that if $f_0(k) = f(0,k)$ is *well behaved*, in that there exist c', k_0 such that $f_0(k) \leq 2^{c'k}$ for all $k \geq k_0$ or $f_0(k) \geq 2^{c'k}$ for all $k \geq k_0$, then it is easy to see that there is some $c'' > 0$ such that $2^{c|\varphi|}f(0,|\varphi|) \leq \max(2^{c''|\varphi|}, c''f(0,|\varphi|))$. Thus, if f_0 is well behaved, then the lower and upper bounds of Theorem 3.4 match, and we have tight bounds in the case of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}^r_{\mathcal{A}}$. This is not the case for $\mathcal{M}^{rt}_{\mathcal{A}}$, $\mathcal{M}^{rst}_{\mathcal{A}}$, and $\mathcal{M}^{elt}_{\mathcal{A}}$, because the sets that arise have exponential-length descriptions.

Do we really have to answer queries of about such complicated formulas if we are to deal with $\mathcal{M}^{rt}_{\mathcal{A}}$, $\mathcal{M}^{rst}_{\mathcal{A}}$, and $\mathcal{M}^{elt}_{\mathcal{A}}$? To some extent, this is an artifact of our insistence that the sets be described using union and set difference. In fact, all the sets that we need to consult the oracle about in our algorithm are atoms, and so have very simple descriptions ($O(|\varphi|)$) if we are allowed to used intersections and complementation. Indeed, suppose that we define an ordering $\prec$ on atoms such that $A_{\mathcal{H}} \prec A_{\mathcal{H}'}$ if $\mathcal{H} \supset \mathcal{H}'$. It follows easily from Lemma 4.3 and Lemma 4.4 that in order to compute $\sigma_2(G)$ (resp., $\sigma_3(G)$, $\sigma_4(G)$), we start by considering all atoms $A_{\mathcal{H}}$ such that G appears positively in $A_{\mathcal{H}}$ and all other sets in $\mathcal{G}_\varphi$ appear negatively; we then need to check whether $|A_{\mathcal{H}}| > 0$ and $|A_{\mathcal{H}}| > 1$ (resp., $|A_{\mathcal{H}}| > 0$, ..., and $|A_{\mathcal{H}}| > |\varphi|$) only for those atoms $A_{\mathcal{H}}$ such that for all $\mathcal{H}' \prec \mathcal{H}$, we have $|A_{\mathcal{H}'}| \leq 1$ (resp., $|A_{\mathcal{H}'}| \leq |\varphi|$). (In addition, in the case of

σ_4, we have also have to check whether $G_1 \cap \ldots \cap G_k = \emptyset$, but again, these are sets with simple descriptions if we allow intersection.) Thus, as long as we can check the required properties of sets described in terms of intersection and complementation relatively efficiently, then the queries to the oracle pose no problem. Unfortunately, the bounds in Proposition 3.3 depends on the descriptions involving only set difference and union, so we cannot get tight bounds for Theorem 3.4 (at least, with our current techniques) using descriptions that involve intersection and complementation. It remains an open question whether we can get tight bounds in all cases taking into account the cost of querying the oracle.

5 Conclusions

We have characterized the complexity of satisfiability for epistemic logics when the set of agents is infinite. Our results emphasize the importance of how the sets of agents are described and provide new information even in the case where the sets involved are finite.

In this paper we have focused on a language that has operators E_G and C_G. There are two interesting directions to consider extending our results.

- We could restrict the language so that it has only E_G operators. If the set of agents is finite (and all sets G are presented in such a way that it is easy to check if $i \in G$), then there are well-known results that show the complexity of the decision problem in this case is PSPACE complete [HM92]. However, again, this result counts E_G as having length $|G|$. Although we have not checked details, it seems relatively straightforward to combine the techniques of [HM92] with those presented here to get PSPACE completeness for $\mathcal{L}_{\mathcal{G}}^{E}$, taking E_G to have length 1, using the same types of oracle calls as in Theorem 3.4. (Note that Proposition 3.3 applies to the language $\mathcal{L}_{\mathcal{G}}^{E}$; we did not use the C_G operators in this proof.)
- We could add the distributed knowledge operator D_G to the language [FHMV95, FHV92, HM92]. Roughly speaking, φ is distributed knowledge if the agents could figure out that φ is true by pooling their knowledge together. Formally, we have

  $$(M, s) \models D_G\varphi \text{ if } (M, t) \models \varphi \text{ for all } t \in \cap_{i\in G}\mathcal{K}_i(s).$$

  It is known that if $\mathcal{A}$ is finite (and there is no difficulty in telling if $i \in G$), then adding D_G to the language poses no essential new difficulties [FHMV95, HM92]. We can get a complete axiomatization, the satisfiability problem for the language with D_G and E_G operators is PSPACE complete, and once we add common knowledge, the satisfiability problem becomes exponential-time complete. Once we allow infinitely many agents, adding D_G introduces new subtleties. For example, even if we place no assumptions on the $\mathcal{K}_i$ relations, once we have both E_G and D_G in the language, we need to be able to distinguish between sets of cardinality one and those with larger cardinality since $E_G p \Leftrightarrow D_G p$ is valid if and only if G is a singleton. New issues also arise once we make further assumptions about the $\mathcal{K}_i$ relations because different properties are preserved for the new agents, say $K_{\mathcal{A}^D}$ and $K_{\mathcal{A}^E}$, which are to be added on as in Proposition 3.5 to represent $D_{\mathcal{A}}$ and $E_{\mathcal{A}}$, respectively. Intuitively, $\mathcal{K}_{\mathcal{A}^E}$ corresponds to the union of the relations K_i

for $i \in G$ while $\mathcal{K}_{\mathcal{A}^D}$ corresponds to their intersection. Thus, while both $K_{\mathcal{A}^D}$ and $K_{\mathcal{A}^E}$ inherit reflexivity and symmetry from the K_i relations, $K_{\mathcal{A}^D}$ inherits transitivity and the Euclidean property while $K_{\mathcal{A}^E}$ does not. There are also additional relations between these agents that must be taken into account. Examples in S4 and S5 include $K_{\mathcal{A}^E}\varphi \Rightarrow K_{\mathcal{A}^D}\varphi$, $K_{\mathcal{A}^E}K_{\mathcal{A}^D}\varphi \Rightarrow K_{\mathcal{A}^E}\varphi$ and $K_{\mathcal{A}^D}K_{\mathcal{A}^E}\varphi \Rightarrow K_{\mathcal{A}^E}\varphi$.

These are issues for future work.

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