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Characterizations of Certain Classes of Norms*

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1. Introduction

A *norm* $\|\cdot\|$ in finite n-dimensional euclidean real or complex space (R^n or C^n) is a real function with the following three properties:

(1.1) $\|x\| > 0$ for all $x \neq 0$, $x \in R^n$ (or C^n).

(1.2) $\|\alpha x\| = \alpha \cdot \|x\|$ for all real numbers $\alpha \geqq 0$.

(1.3) $\|x + y\| \leqq \|x\| + \|y\|$ for all $x, y \in R^n$ (or C^n).

The norm $\|\cdot\|^D$ (defined in the space of all row vectors y^H) *dual* to the norm $\|\cdot\|$ is defined by

(1.4) $$\|y^H\|^D := \max_{x \neq 0} \frac{\operatorname{Re} y^H x}{\|x\|}.$$

One important class of norms is the *strictly homogenous* norms, that is, norms defined in R^n with the property

(1.5) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all real numbers α,

and norms in C^n with the property

(1.6) $\|\beta x\| = |\beta| \cdot \|x\|$ for all complex numbers β.

Another increasingly important class of norms is the class of *absolute* norms. A norm $\|\cdot\|$ is called absolute if

(1.7) $\|x\| = \||x|\|$[1] for all x.

Absolute norms have the following two equivalent characterizations, first proved by Bauer, Stoer and Witzgall in [3], which are important in the study of exclusion and inclusion theorems for eigenvalues of a matrix ([2]):

(1.8) $|x| \leqq |y|$[2] implies $\|x\| \leqq \|y\|$ (monotonic);

(1.9) $\operatorname{lub}(D) = \max(d_{ii})$ for all diagonal matrices D (axis-oriented);

where $\operatorname{lub}(A)$ is the *least upper bound norm* of an $n \times n$ matrix A with respect to the norm $\|\cdot\|$:

(1.10) $$\operatorname{lub}(A) := \max_{x \neq 0} \frac{\|A x\|}{\|x\|}.$$

[1] If $x = (x_1, \ldots, x_n)^T$, then $|x| = (|x_1|, \ldots, |x_n|)^T$.
[2] $x \leqq y$ means $x_i \leqq y_i$, $i = 1, \ldots, n$.

One purpose of this paper is to provide new characterizations of these two norm classes. In section 3 it will be shown that a strictly homogenous norm in C^n or an absolute norm in C^n may be identified by properties of *dual vector pairs*; a pair of non-zero vectors y^H, x is called *dual*, written $y^H \| x$, if

$$\mathrm{Re}\, y^H x = \|y^H\|^D \|x\|. \tag{1.11}$$

Geometrically (see for example [6, 7, 9]), $y^H \| x$ if and only if y is the normal to a *support hyperplane H*

$$H := \{x \mid \mathrm{Re}\, y^H x = \|y^H\|^D\}$$

to the compact convex body B

$$B := \{x \mid \|x\| \leqq 1\}$$

through the point $x/\|x\|$ (Fig. 1).

In section 4 the absolute norms in C^n will be characterized with the aid of the BAUER [1] *field of values* $G[A]$ *of a matrix* A *with respect to a norm* $\|\cdot\|$,

$$G[A] := \{y^H A x \mid y^H \| x,\ \|y^H\|^D = \|x\| = 1\}; \tag{1.12}$$

a necessary and sufficient condition for a norm $\|\cdot\|$ in C^n to be absolute is

$$D = \mathrm{diag}(d_{11}, \ldots, d_{nn}) \text{ implies } G[D] = \mathscr{H}^{3}\{d_{11}, \ldots, d_{nn}\}. \tag{1.13}$$

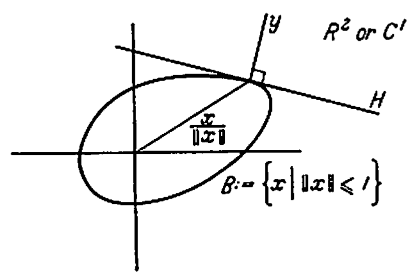

Fig. 1

In [13], STOER and WITZGALL proved the following

(1.14) **Theorem.** *Let* $\|:\|$ *be an absolute norm and* $x > 0$, $y > 0$ *any two non-zero vectors. Then there exists a unique (up to positive multiples) non-singular non-negative diagonal matrix* $D \geqq 0$ *such that*

$$y^H D \| D^{-1} x. \tag{1.15}$$

In [8], in order to extend theorem (1.14) (which is used in this paper) to other norms, a new class of norms, the *orthant-monotonic* norms, was introduced. Section 2 is devoted to an investigation of the properties of these norms. It turns out that this class, especially in R^n, possesses some interesting properties. For instance, (1.13) is a characterization not of the absolute norms in R^n but of the orthant-monotonic norms in R^n.

In section 5 we give a list of known characterizations of the classes of strictly homogenous, absolute, and orthant-monotonic norms.

2. Orthant-Monotonic Norms

In order to define this class of norms we need the following correspondence between C^n and R^{2n}. If

$$x = \begin{pmatrix} \vdots \\ x_j' + i x_j'' \\ \vdots \end{pmatrix}, \quad x_j', x_j'' \text{ real}$$

[3] $\mathscr{H}\{M\}$ denotes the convex hull of the set M.

is a vector in C^n, then the vector x^R in R^{2n} is defined by

$$(2.1)\qquad x^R := \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \\ x''_1 \\ \vdots \\ x''_n \end{pmatrix}$$

In other words, if $x = x' + i x''$ where x' and x'' are real vectors, then $x^R := x' \oplus x''$.

Given a norm $\|\cdot\|$ defined in C^n we can then define the function $\|\cdot\|_R$ in R^{2n} by

$$(2.2)\qquad \|x^R\|_R := \|x\| \quad \text{for all vectors } x^R \text{ of } R^{2n}.$$

$\|\cdot\|_R$ is obviously a norm. We then have the following

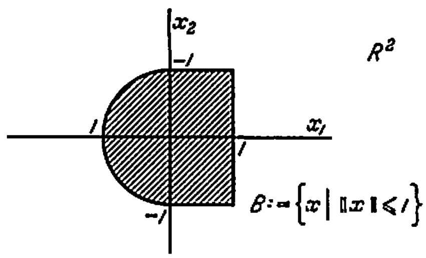

Fig. 2

(2.3) **Definition.** *A norm* $\|\cdot\|$ *in* R^n *is orthant-monotonic* if for all vectors x, y

$$(2.4)\qquad \begin{array}{l} |x| \leqq |y| \quad \text{and} \quad y_j x_j \geqq 0, \\ j = 1, \ldots, n \quad \text{implies} \quad \|x\| \leqq \|y\|. \end{array}$$

A norm $\|\cdot\|$ *in* C^n *is orthant-monotonic* if and only if the corresponding norm $\|\cdot\|_R$ in R^{2n} is orthant-monotonic.

It is easy to see that definition (2.3) is weaker than the Bauer-Stoer-Witzgall definition of monotonic norms ((1.8)). All absolute norms are orthant-monotonic, but the following orthant-monotonic norm in R^2 is *not* absolute (see also Fig. 2).

$$(2.5)\qquad \|x\| := \begin{cases} \|x\|_\infty = \max(|x_1|, |x_2|) & \text{if } x_1 \geqq 0 \\ \|x\|_2 = \sqrt{x_1^2 + x_2^2} & \text{if } x_1 \leqq 0. \end{cases}$$

Definition (2.3) is also different from SALLIN's definition of monotonicity ([*12*]): $|y| \leqq x$ implies $\|y\| \leqq \|x\|$.

In order to investigate the properties of this new class, we need the following definitions. Given an n-tuple ξ,

$$(2.6)\qquad \xi := (v_1, \ldots, v_n) \quad \text{where} \quad v_j = \pm 1, \qquad j = 1, \ldots, n,$$

we define the ξ-*orthant* in R^n as the set

$$(2.7)\qquad R^n_\xi := \{x \in R^n \mid x_j v_j \geqq 0,\ j = 1, \ldots, n\}.$$

If ξ is a $2 \times n$ tuple,

$$(2.8)\qquad \xi := (v_1, \ldots, v_{2n}) \quad \text{where} \quad v_j = \pm 1, \quad j = 1, \ldots, 2n,$$

we define the ξ-*orthant* in C^n as the set

$$(2.9)\qquad C^n_\xi := \{x \in C^n \mid x^R \in R^{2n}_\xi\}.$$

We then use $\Xi(x)$ to denote the set of all ξ-orthants to which x belongs:

$$(2.10)\qquad \Xi(x) := \{R^n_\xi \text{ (or } C^n_\xi) \mid x \in R^n_\xi \text{ (or } C^n_\xi)\}.$$

A vector x belongs to more than one orthant only if a component of x, or the real or imaginary part of a component, vanishes for $x \in R^n$, or $x \in C^n$ resp.

(2.11) **Definition.** Let ξ be as in (2.6). *A norm* $\|\cdot\|$ *defined in* R^n *is monotonic in the* ξ*-orthant, or* R^n_ξ*-monotonic,* if

$$x, y \in R^n_\xi, \qquad |x| \leqq |y| \quad \text{implies} \quad \|x\| \leqq \|y\|.$$

Given ξ as in (2.8), a *norm* $\|\cdot\|$ *defined in* C^n *is called monotonic in the* ξ*-orthant,* or C^n_ξ*-monotonic,* if

$$x, y \in C^n, \qquad |x^R| \leqq |y^R| \quad \text{implies} \quad \|x\| \leqq \|y\|.$$

As a consequence of definition (2.11) we have obviously

(2.12) **Lemma.** *A norm is orthant-monotonic if and only if it is monotonic in every orthant.*

From definition (2.3) we have the fact that $\|\cdot\|$ in C^n is orthant-monotonic if and only if the corresponding norm $\|\cdot\|$ in R^{2n} is orthant-monotonic. This fact makes it easier to characterize these norms, since in some cases we may restrict the proofs to the real case (R^n). The following lemma will be of help.

(2.13) **Lemma.** *Let* $\|\cdot\|$ *be a norm in* C^n *and* $\|\cdot\|_R$ *the corresponding norm in* R^{2n} (see (2.2)). *Then*

$$y^H \| x \quad \text{if and only if} \quad (y^R)^H \|_R\, x^R,$$

where $\|_R$ *means dual with respect to the norm* $\|\cdot\|_R$.

Proof. For $y, x \in C^n$, using the notation of (2.1), we have

$$\operatorname{Re} y^H x = \sum_{j=1}^{n} (y'_j x'_j + y''_j x''_j) = (y^R)^H x^R. \tag{2.14}$$

Therefore the following holds for $\|\cdot\|_R^D$, the dual of $\|\cdot\|_R$,

$$\|(y^R)^H\|_R^D = \max_{x^R \neq 0} \frac{(y^R)^H x^R}{\|x^R\|_R} = \max_{x \neq 0} \operatorname{Re} \frac{y^H x}{\|x\|} = \|y^H\|^D. \tag{2.15}$$

The proof follows from (2.14), (2.15), and (2.3) and from the definition of duality:

$$\operatorname{Re} y^H x = \|y^H\|^D \|x\|.$$

We are now ready to characterize norms monotonic in one orthant, R^n_ξ. A theorem corresponding to (2.16), but for absolute norms, was proven in [3].

(2.16) **Theorem.** *Let* $\|\cdot\|$ *be a norm in* R^n *and let* $\xi = (v_1, \ldots, v_n)$ *where* $v_j = \pm 1$, $j = 1, \ldots, n$. $\|\cdot\|$ *is* R^n_ξ*-monotonic if and only if the positive definite and homogenous function*

$$f(x) := \|x(\xi)\|,$$

where $x(\xi)$ *is the vector with components* $v_j |x_j|$, *is subadditive — that is when* f *is a norm.*

Note. The same theorem holds also in C^n with a $2n$-tuple and $x(\xi)$ defined accordingly.

Proof. Suppose $\|\cdot\|$ is R^n_ξ-monotonic. We have only to prove the inequality

$$f(x+y) \leqq f(x) + f(y).$$

Now $x(\xi), y(\xi) \in R^n_\xi$ implies

$$|(x+y)(\xi)| \leqq |x(\xi) + y(\xi)| = |x(\xi)| + |y(\xi)|.$$

From the monotonicity in R^n_ξ follows $\|(x+y)(\xi)\| \leqq \|x(\xi)\| + \|y(\xi)\|$. Therefore, it follows that

$$f(x+y) = \|(x+y)(\xi)\| \leqq \|x(\xi)\| + \|y(\xi)\| = f(x) + f(y).$$

Now suppose that $f(x) = \|x(\xi)\|$ is a norm. Then $f(|x|) = f(x(\xi)) = f(x)$ and f is by definition an absolute norm. Therefore f is also orthant-monotonic.

The following theorem provides a basis for a characterization of orthant-monotonic norms.

(2.17) **Lemma.** $\|\cdot\|$ *is monotonic in a ξ-orthant if and only if*

(2.18) $$x \in \xi\text{-orthant}, \quad y^H \| x \quad \text{implies} \quad y \in \Xi(x).$$

Proof. Because of lemma (2.13) we may restrict the proof to norms in R^n. Let $\|\cdot\|$ be R^n_ξ-monotonic and let $x \in R^n$. From $y^H\|x$ follows

$$\|y^H\|^D = \frac{y^H x}{\|x\|} = \frac{\sum y_j x_j}{\|x\|} = \frac{\sum\limits_{j \neq k} y_j x_j + y_k x_k}{\|x\|} = \frac{p_k + y_k x_k}{\|x\|} \quad \text{for} \quad 1 \leqq k \leqq n.$$

Suppose $y_k x_k < 0$ for one k. Then $p_k > 0$. For the vector

$$z = (x_1, \ldots, x_{k-1}, \tfrac{1}{2} x_k, x_{k+1}, \ldots, x_n)^T$$

we have $x, z \in R^n_\xi$, $|z| \leqq |x|$, and from the R^n_ξ-monotonicity follows

$$\|z\| \leqq \|x\|.$$

We then have the following contradiction:

$$\|y^H\|^D = \frac{p_k + y_k x_k}{\|x\|} < \frac{p_k + \frac{1}{2} y_k x_k}{\|x\|} \leqq \frac{y^H z}{\|z\|} \leqq \|y^H\|^D.$$

Therefore $y_k x_k \geqq 0$ for all k and $y \in \Xi(x)$. To prove the sufficiency of (2.18), we take any two vectors μ, $x \in R^n$ with $|\mu| \leqq |x|$ and $\mu \neq 0$. Let us first assume that $\mu_j \neq 0$ in case $x_j \neq 0$. Then there exists a $y \in \Xi(\mu)$ with $y^H \| \mu$, yielding

$$\|\mu\| = \frac{y^H \mu}{\|y^H\|^D} = \frac{\sum y_j \mu_j}{\|y^H\|^D} \leqq \frac{\sum y_j x_j}{\|y^H\|^D} = \frac{y^H x}{\|y^H\|^D} \leqq \|x\|.$$

Because the norm $\|\cdot\|$ is a continuous function, $\|\mu\| \leqq \|x\|$ must also hold if a component $\mu_j = 0$ and $x_j \neq 0$. The theorem is thus proved.

From lemma (2.17) we get immediately the following two results.

(2.19) **Corollary.** $\|\cdot\|$ *is orthant-monotonic if and only if*

$$y^H \| x \quad \text{implies} \quad y \in \Xi(x).$$

(2.20) **Corollary.** *A norm $\|\cdot\|$ defined in R^n is orthant-monotonic if and only if*

(2.21) $$y^H \| x \quad \text{implies} \quad (y_j x_j \geqq 0, \; j = 1, \ldots, n).$$

It is interesting to note that in C^n the absolute norms instead of the orthant-monotonic norms will be characterized by (2.21) (see section 3). We can now use these corollaries to extend a result of NIRSCHL and SCHNEIDER ([9]).

(2.22) **Theorem.** *Let* $\|\cdot\|$ *be orthant-monotonic and let* $x \geqq 0$, $z \geqq 0$ *be two vectors with the properties*

$$0 \leqq x \leqq z \quad \text{and} \quad \|x\| = \|z\|.$$

Then for each y^H *dual to* x *we have*

$$x_j < z_j \quad \textit{implies} \quad \operatorname{Re} y_j = 0.$$

Proof. Suppose that for some j with $x_j < z_j$ there exists a vector y, $y^H \| x$, with $\operatorname{Re} y_j \neq 0$. Using (2.19) we then have the contradiction

$$\|z\| = \|x\| = \frac{\operatorname{Re} y^H x}{\|y^H\|^D} < \frac{\operatorname{Re} y^H z}{\|y^H\|^D} \leqq \|z\|.$$

With the aid of corollary (2.19) we obviously also have

(2.23) **Theorem.** *A norm* $\|\cdot\|$ *is orthant-monotonic if and only if the dual norm* $\|\cdot\|^D$ *is orthant-monotonic.*

We can obtain a last characterization of the orthant-monotonic norms using the concept of a norm $\|\cdot\|_L$, defined on a subspace L or C^n or R^n, induced by the norm $\|\cdot\|$ in C^n or R^n by setting

$$\|x\|_L := \|x\| \quad \text{for} \quad x \in L.$$

By a coordinate subspace $L = V_\eta$ of C^n or R^n we mean a subspace spanned by some subset $\{e_i | i \in N = \{1, 2, \ldots, n\}\}$ of the set of all axis vectors $e_1 := (1, 0, \ldots, 0)^T$, $\ldots, e_n = (0, \ldots, 0, 1)^T$. Then every $x \in R^n$ (or C^n) can be written in the form

$$x = x_\eta \oplus x_{\eta'} \quad \text{with} \quad x_\eta \in V_\eta, \quad x_{\eta'} \in V_{\eta'}, \quad \eta' := N \backslash \eta.$$

Obviously,

(2.24) **Lemma.** *If* $\|\cdot\|_\eta$ *is defined by* $\|x_\eta\|_\eta := \|x\|$ *for* $x = x_\eta \oplus 0_{\eta'}$; *and if* $\|\cdot\|$ *is orthant-monotonic, then* $\|\cdot\|_\eta$ is orthant-monotonic in V_η.

In [8] the following was proved,

(2.25) **Lemma.** *Let* $\|\cdot\|$ *be an orthant-monotonic norm in* R^n *or* C^n *and* $\eta \subseteq N = \{1, 2, \ldots, n\}$, $\eta' := N \backslash \eta$. *Then the following holds:*

$$(\|y_\eta^H\|_\eta)^D = (\|y_\eta^H\|^D)_\eta.$$

That is, the dual of the induced norm is the same as the norm induced by the dual of $\|\cdot\|$. We can use this lemma to help prove the following

(2.26) **Characterization.** *Let* $\|\cdot\|$ *be a norm in* R^n, $n > 1$, *and let* $\|\cdot\|_j$ *be the norm induced by* $\|\cdot\|$ *in the subspace* η *of the axis vectors* $e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$. *Then* $\|\cdot\|$ *is orthant-monotonic if and only if*

(2.27) $$(\|y_\eta^H\|_j)^D = (\|y_\eta^H\|^D)_j \quad \text{for} \quad j = 1, \ldots, n, \quad \text{all } y_\eta^H \in R^{n-1},$$

Proof. The necessity of (2.27) is lemma (2.25). Suppose on the other hand that (2.27) holds. Then for each $x_\eta \in R^{n-1}$ and a fixed j we have

$$\|x_\eta \oplus 0_{\eta'}\| = \|x_\eta\|_j = \sup_{y_\eta \neq 0} \frac{y_\eta^H x_\eta}{\|y_\eta^H\|_j^D} = \sup_{\substack{y_\eta \oplus 0 \\ y_\eta \neq 0}} \frac{(y_\eta \oplus 0)^H (x_\eta \oplus \alpha)}{\|(y_\eta \oplus 0)^H\|^D} \leqq \|x_\eta \oplus \alpha\| \quad \text{for all real } \alpha.$$

Therefore, for any pair $x\in R^n$, $z\in R^n$ with

$$x=x_n\oplus\alpha,\quad z\in\Xi(x),\quad |z|\leqq|x|,\quad |z_j|<|x_j| \text{ for one } j,\ z_i=x_i \text{ for } i\neq j$$

we have $\|z\|\leqq\|x\|$. Indeed, there exists a λ, $1\geqq\lambda\geqq 0$, such that

$$z=\lambda(x_n\oplus 0)+(1-\lambda)\,x$$

and therefore

$$\|z\|=\|\lambda(x_n\oplus 0)+(1-\lambda)\,x\|\leqq\lambda\|x_n\oplus 0\|+(1-\lambda)\,\|x\|\leqq\lambda\|x\|+(1-\lambda)\,\|x\|=\|x\|.$$

In case that z and x differ in more than one component, but $x, z\in\Xi(x)$ and $|z|\leqq|x|$, the proof follows by induction on the number of indices k such that $|z_k|<|x_k|$.

3. Characterization by Dual Vector Pairs

In this section we want to characterize absolute norms and strictly homogenous norms in C^n by properties of dual pairs of vectors. For both characterization we need the following

(3.1) **Lemma.** *Let M be a convex compact set in R^2 with inner points. If every non zero vector $x\in \mathrm{Rd}\,M$*[4] *is itself normal to a support hyperplane (here a line) of the set M through x:*

$$(3.2)\qquad x^H z\leqq x^H x \quad\text{for all}\quad z\in M,$$

then M is a circular disk with 0 as center.

Proof. We first prove that 0 is an inner point of M. If $0\notin M$, then there exists a line passing through 0 and an inner point of M which intersects $\mathrm{Rd}\,M$ in two points y and ϱy with $\varrho>1$. Therefore

$$y^H(\varrho y)>y^H y,\qquad \varrho y\in M,\qquad y\in\mathrm{Rd}\,M,$$

in contradiction to (3.2). Suppose $0\in\mathrm{Rd}\,M$. Without loss of generality assume that the line $x=0$ is a support hyperplane to M through 0, and that there exists a point $y>0$ in M. Because M is convex, there must exist a point $x\neq 0$ in $\mathrm{Rd}\,M$ with $x\neq y$ but $0\leqq x\leqq y$. Therefore

$$x^H y>x^H x,\qquad y\in M,\qquad x\in\mathrm{Rd}\,M$$

contradicting (3.2). Therefore 0 is an inner point of M. This means that M *defines a norm* $\|\cdot\|$ (see [6, 14]) by

$$\|x\|:=\inf\{\omega\geqq 0\,|\,x\in\omega M\}\quad\text{where}\quad \omega M:=\{\omega x\,|\,x\in M\}$$

with the property

$$M:=\{x\,|\,\|x\|=1\}.$$

Property (3.2) can now be interpreted to mean that $x^H\|x$ for all $x\in\mathrm{Rd}\,M$.

We proceed by showing that if $x\neq 0$, then x^H is the only vector dual to x, i.e. $\mathrm{Rd}\,M$ is a differentiable curve (see [6]). Suppose $x\neq 0$, $x\in\mathrm{Rd}\,M$ and $w^H\|x$, $\|w\|=1$ and $w\neq x$. By hypothesis $w^H\|w$. Because of the convexity of the set

$$\{z\,|\,w^H\|z,\ \|z\|=1\}$$

[4] $\mathrm{Rd}\,M$ denotes the boundary of M.

(see for instance BAUER [1]) we have $\|z\|=1$, $w^H\|z$ for all z in the set Z defined by

$$Z := \{\lambda x + (1-\lambda) w \mid 0 < \lambda < 1\}.$$

The points $z \in Z$ lie therefore on the support line to M defined by w and can have no other dual vector than w, in contradiction to $z^H\|z$ (hypothesis).

To complete the proof, we write the coordinates (x_1, x_2) of the differentiable curve $\operatorname{Rd} M$ as differentiable functions of the length S of the curve: $x_1 = x_1(S)$, $x_2 = x_2(S)$. From $x^H\|x$ for $x \in \operatorname{Rd} M$ we immediately infer that the direction of the tangent to $\operatorname{Rd} M$ at x, $(\dot{x}_1, \dot{x}_2)$, is perpendicular to $x = (x_1, x_2)$, i.e.

$$0 = x_1 \dot{x}_1 + x_2 \dot{x}_2 = \frac{1}{2} \frac{d(x_1^2 + x_2^2)}{dS}.$$

$x_1^2 + x_2^2$ is therefore constant and $\operatorname{Rd} M$ is a circle about the 0 point. QED.

We are now ready to prove the following

(3.3) **Characterization.** *A norm* $\|\cdot\|$ *in* C^n *is absolute if and only if*

(3.4) $$y^H\|x \quad \text{implies} \quad (\bar{y}_k x_k \geqq 0,\ k = 1, \ldots, n).$$

Proof. The necessity was first proved by NIRSCHL and SCHNEIDER [9] and is not given here. To prove that (3.4) is a sufficient condition for $\|\cdot\|$ to be absolute, it is necessary to show that

(3.5) $$\|x\| = \|x(\vartheta, j)\| \quad \text{for,} \quad 0 \leqq \vartheta \leqq 2\pi, \qquad j = 1, \ldots, n,$$

where the vector $x(\vartheta, j)$ is defined by

$$x(\vartheta, j) := \begin{pmatrix} x_1 \\ \vdots \\ x_{j-1} \\ e^{i\vartheta} x_j \\ x_{j+1} \\ \vdots \\ x_n \end{pmatrix}.$$

The sufficiency is obvious if (3.5) is true, since then the norm depends only on the absolute value of the components. Because norms are homogenous (property (1.2)) we may assume $\|x\| < 1$.

Let therefore x be any fixed vector and j any fixed index, $1 \leqq j \leqq n$, with the properties

(3.6) $$\|x\| < 1, \qquad x_j \neq 0.$$

Since $\|x\| < 1$, the plane E

(3.7) $$E := \{x + \alpha e_j \mid \alpha \text{ complex}\} \quad \text{where } e_j \text{ is the } j^{\text{th}} \text{ axis vector},$$

is certainly *not* contained in a support hyperplane to the convex compact body

$$B := \{z \mid \|z\| \leqq 1\}.$$

This means that

(3.8) $$z \in \operatorname{Rd}(B \cap E), \qquad y^H\|z \text{ implies } y_j \neq 0,$$

for if y_j were zero, then from the definition of E and the fact that $z\in E$, $x\in E$, we would have

$$\|z\|\cdot\|y^H\|^D = \operatorname{Re} y^H z = \operatorname{Re} y^H x \leqq \|x\|\cdot\|y^H\|^D,$$

leading to a contradiction of (3.6):

$$1 = \|z\| \leqq \|x\|.$$

Property (3.5) is equivalent to the condition that the non-empty compact convex set $M\subset C^1$

$$M := \{z_j \mid z\in E,\ \|z\|\leqq 1\} = \{z_j \mid z\in E\cap B\} \tag{3.9}$$

be a circular disk with 0 as center. This we now prove using lemma (3.1). In order to use the lemma, given any point $z_j \neq 0$ in $\operatorname{Rd} M$ we must show that the (one dimensional) vector z_j itself is a normal to a support hyperplane of M through z_j, i.e.

$$\operatorname{Re} \bar z_j w \leqq \bar z_j z \quad \text{for all} \quad w\in M. \tag{3.10}$$

Given $z_j \neq 0$ in $\operatorname{Rd} M$ we form the vector z (using our fixed vector x) by

$$z^H = (\bar x_1, \ldots, \bar x_{j-1}, \bar z_j, \bar x_{j+1}, \ldots, \bar x_n).$$

Obviously, from the definition of M, B and E, we have $z\in \operatorname{Rd} B\cap E$. There exists a vector y^H with $y^H\|z$. From (3.4) (our assumption) and (3.8) follows $\bar y_j z_j > 0$ and therefore

$$y_j = \varrho z_j \quad \text{with} \quad \varrho > 0. \tag{3.11}$$

Since $y^H\|z$, we have for all $w\in B\cap E$

$$\operatorname{Re} y^H w \leqq \operatorname{Re} y^H z.$$

From the definition of M follows (since w, z are in E)

$$\operatorname{Re} \bar y_j w_j \leqq \operatorname{Re} \bar y_j z_j \quad \text{for all} \quad w_j \in M.$$

Finally, from (3.11) we have

$$\operatorname{Re} \bar z_j w_j \leqq \bar z_j z_j \quad \text{for all} \quad w_j \in M$$

which was what we wanted to prove ((3.10)). The hypothesis of lemma (3.1) is therefore fulfilled (considering M as a set in R^2 instead of C^1) and the theorem is proved.

Statement (3.3) should be compared with the corresponding statement (2.20) in the case of R^n. We now prove the analogous case of strictly homogenous norms. The proof is quite similar to the proof of (3.3). The necessity of (3.12) was first proved by BAUER [*1*].

(3.12) **Characterization.** *A norm* $\|\cdot\|$ *in* C^n *is strictly homogenous if and only if*

$$y^H\|x \quad \text{implies} \quad y^H x > 0. \tag{3.13}$$

Proof. Suppose $\|\cdot\|$ is strictly homogenous. Then for dual pairs y^H, x we have

$$\operatorname{Re} y^H x = \|y^H\|^D \|x\| = \|y^H\|^D \|e^{i\vartheta} x\| \geqq \operatorname{Re} e^{i\vartheta} y^H x \quad \text{for all } \vartheta.$$

From this follows

$$\operatorname{Re} y^H x = |y^H x| \quad \text{and} \quad y^H x > 0.$$

Now suppose that (3.13) holds and let $x \neq 0$ be any fixed vector. We must show that

$$\|x\| = \|e^{i\vartheta} x\| \quad \text{for all } \vartheta. \tag{3.14}$$

We first define the plane E of C^n by

$$E := \{\varrho e^{i\vartheta} x \mid \varrho > 0,\ 0 \leqq \vartheta \leqq 2\pi\}.$$

Now (3.14) is certainly true if the non-empty convex compact set M in C^1 with inner points,

$$M := \{\varrho e^{i\vartheta} \mid \varrho \geqq 0,\ 0 \leqq \vartheta \leqq 2\pi,\ \|\varrho e^{i\vartheta} x\| \leqq 1\},$$

is a circular disk with 0 as the center. This we now prove. Let $\varrho_1 e^{i\vartheta_1}$ be on Rd M Then the vector

$$z = \varrho_1 e^{i\vartheta_1} x \quad \text{is in Rd } B, \text{ where as usual} \quad B = \{x \mid \|x\| \leqq 1\}. \tag{3.15}$$

We pick a vector y^H dual to z and write it in the form

$$y = y_p + y_N = \varrho_2 e^{i\vartheta_2} x + y_N \qquad \varrho_2 > 0, \qquad 0 \leqq \vartheta_2 \leqq 2\pi,$$

where y_p is the orthogonal projection of y onto the plane E: $y_N^H z = 0$. Using (3.13) we have

$$y^H z = y_p^H z = \varrho_1 \varrho_2 e^{i(\vartheta_1 - \vartheta_2)} x^H x > 0.$$

Therefore, $\vartheta_1 = \vartheta_2$ and

$$y_p = \varrho_2 e^{i\vartheta_1} x, \qquad z = \varrho_1 e^{i\vartheta_1} x, \quad \text{where} \quad \varrho_1 > 0, \qquad \varrho_2 > 0.$$

i.e. y_p is a positive multiple of z. Furthermore, for $\alpha = \varrho e^{i\vartheta} \in M$ $(\varrho \neq 0)$ we have also $\alpha x \in B$, and therefore

$$\operatorname{Re} y_p^H (\alpha x) = \operatorname{Re} \varrho \varrho_2 e^{i(\vartheta - \vartheta_1)} x^H x \leqq y_p^H z = \varrho_1 \varrho_2 x^H x.$$

Multiplying the last line by $\varrho_1/(\varrho_2 x^H x)$ we have

$$\operatorname{Re} \overline{(\varrho_1 e^{i\vartheta_1})} (\varrho e^{i\vartheta}) \leqq \overline{(\varrho_1 e^{i\vartheta_1})} (\varrho_1 e^{i\vartheta_1}), \quad \text{for all} \quad \varrho e^{i\vartheta} \in M.$$

In other words, the non-zero vector $\varrho_1 e^{i\vartheta_1} \in \text{Rd}\, M$ is normal to a support hyperplane of M through the point $\varrho_1 e^{i\vartheta_1}$. We now apply lemma (3.1) to M (interpreting it as a set in R^2 instead of C^1) and the theorem is proved.

The following theorem should be compared with an analogous theorem for absolute norms, proved in [3],

(3.16) **Characterization.** *A norm $\|\cdot\|$ is strictly homogenous if and only if*

$$\|x\| \cdot \|y^H\|^D \geqq |y^H x| \quad \text{for all } y, x. \tag{3.17}$$

Proof. From (3.17) and

$$\|e^{i\vartheta} x\| = \max_{y \neq 0} \frac{e^{i\vartheta} y^H x}{\|y^H\|^D} \leqq \max_{y \neq 0} \frac{|y^H x|}{\|y^H\|^D}$$

we deduce

$$\|e^{i\vartheta} x\| = \max_{y \neq 0} \frac{|y^H x|}{\|y^H\|^D}.$$

Therefore $\|e^{i\vartheta}x\|$ does not depend on ϑ. Suppose now that $\|\cdot\|$ is strictly homogenous. Then

$$\|x\|\cdot\|y^H\|^D = \|e^{i\vartheta}x\|\|y^H\|_D \geqq \operatorname{Re} e^{i\vartheta} y^H x \quad \text{for all } \vartheta$$

and (3.17) follows.

4. Characterization by Fields of Values

In [8] the following extension of theorem (1.14) was proved

(4.1) **Theorem.** *Let $\|\cdot\|$ be any norm in R^n or an orthant-monotonic norm in C^n. Let $x>0$, $y>0$ be two vectors with positive components. Then there exists one (and up to positive multiples only one) non-singular non-negative diagonal matrix $D\geqq 0$ such that*

(4.2) $$y^H D \| D^{-1} x.$$

Following a suggestion of STOER, we now use this theorem to prove the following

(4.3) **Lemma.** *For all norms in R^n and all orthant-monotonic norms in C^n the following holds for all diagonal matrices $D=\operatorname{diag}(d_1,\ldots,d_n)$:*

$$G[D] \supset \mathscr{H}(d_1,\ldots,d_n).$$

Proof. Let $\alpha=\sum\lambda_j d_j$ where $1\geqq\lambda_j>0$, $\sum\lambda_j=1$. By theorem (4.4), for the two vectors

$$x=\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix} \quad\text{and}\quad y=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

there exists a diagonal matrix D_0 such that

$$y^H D_0 \| D_0^{-1} x.$$

Therefore

$$\alpha=\sum\lambda_j d_j = y^H D x = y^H D D_0 D_0^{-1} x = y^H D_0 D D_0^{-1} x \in G[D].$$

Because $G[D]$ is compact (see [1]), every point

$$\alpha=\sum\lambda_j d_j,\qquad \lambda_j\geqq 0,\qquad \sum\lambda_j=1$$

in $\mathscr{H}(d_1,\ldots,d_n)$ also lies in $G[D]$. We now prove the main theorem of this section.

(4.4) **Theorem.** *$\|\cdot\|$ in C^n is absolute if and only if $D=\operatorname{diag}(d_1,\ldots,d_n)$ implies $G[D]=\mathscr{H}\{d_1,\ldots,d_n\}$. $\|\cdot\|$ in R^n is orthant-monotonic if and only if*

$$D=\operatorname{diag}(d_1,\ldots,d_n) \quad\text{implies}\quad G[D]=\mathscr{H}\{d_1,\ldots,d_n\}.$$

Proof. According to Lemma (4.3) $G[D]\supset\mathscr{H}\{d_j\}$ for these norms. We need therefore only show that $G[D]\subset\mathscr{H}\{d_1,\ldots,d_n\}$. Let $\|\cdot\|$ be one of the above norms and let $\alpha\in G[D]$. Then

$$\alpha=y^H D x. \qquad \operatorname{Re} y^H x = \|y^H\|^D\|x\| = 1$$

for some vectors y, x.

For $\|\cdot\|$, it then follows from (3.3) and (2.20) that $\bar y_j x_j \geqq 0$, $j=1,\ldots,n$. Therefore

$$\alpha=\sum_{j=1}^{n}(y_j x_j)\,d_j,\qquad (\bar y_j x_j)\geqq 0,\qquad j=1,\ldots,n,\qquad \sum\bar y_j x_j = 1.$$

In other words, $\alpha\in\mathscr{H}\{d_1,\ldots,d_n\}$ and the theorem is proved.

5. Table of Characterizations

The following table lists all known (to the author) characterizations of the three main classes of norms mentioned in this paper. In the table, $D = \text{diag}(d_{11}, \dots, d_{nn})$ is a diagonal matrix; $\Xi(x)$, defined in section 2, is the set of orthants to which x belongs. The first three characterizations were first proved in [3]. Particularly interesting is the relationship between the properties of absolute and strictly homogenous norms — No. 2 and No. 7, and No. 4 and No. 8.

Table

Property	Class of norms characterized in	
	C^n	R^n
1. $\lvert x\rvert < \lvert y\rvert$ implies $\|x\| \leq \|y\|$	absolute	absolute
2. $\|y^H\|^D \|x\| \geqq \lvert y\rvert^H \lvert x\rvert$	absolute	absolute
3. $\text{lub}(D) = \max_i \lvert d_{ii}\rvert$	absolute	absolute
4. $y^H \| x$ implies $(\bar{y}_j x_j \geqq 0, j = 1, \dots, n)$	absolute	orthant-monotonic
5. $G[D] = \mathscr{H}\{d_{ii}\}$	absolute	orthant-monotonic
6. $y^H \| x$ implies $y \in \Xi(x)$	orthant-monotonic	orthant-monotonic
7. $\|y^H\|^D \|x\| \geqq \lvert y^H x\rvert$	strictly homogenous	strictly homogenous
8. $y^H \| x$ implies $y^H x > 0$	strictly homogenous	all norms

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