Algorithmic aspects of communication networks	**Lecture 16**
OR 739 Spring 1997	**February 26, 1997**

We would like to state a lemma as follows: Assume $X_1, \ldots, X_n$ are independent random variables such that $\sum_i B_r(X_i) \geq 4k$ where $r = \frac{2\log(p^{-1}L)}{k}$. Then

$$\Pr(\sum_i X_i > k) > p.$$

However, we need more assumptions than we make here to prove this lemma. Therefore, we restrict X_i's such that

- X_i's are on-off random variables
- $\Pr(X_i = z_i) = q_i$ and $\Pr(X_i = 0) = 1 - q_i$
- $z_i \leq r^{-1}$

Claim 1 *$r^{-1}q(e^{rz} - 1) \leq qze^{rz}$ where $z, q, r \geq 0$.*

Proof. Consider both sides of the inequality as function of z. For $z = 0$, equality holds. If we take the derivatives of both sides, we see that the derivative of left-hand side(qe^{rz}) is less than the derivative of the right-hand side ($qe^{rz} + qzre^{rz}$). Therefore, inequality holds. ■

Lemma 1 *If X is an on-off random variable and $X \leq r^{-1}$ then*

$$B_r(X) \leq e\mathrm{E}(X).$$

Proof.

$$\begin{aligned} B_r(X) &= r^{-1}\log \mathrm{E}(e^{rX}) \\ &= r^{-1}\log(qe^{rz} + (1-q)e^0) \\ &= r^{-1}\log(q(e^{rz} - 1) + 1) \\ &\leq r^{-1}q(e^{rz} - 1) \text{ by } \log(1+x) \leq x \\ &\leq qze^{rz} \text{ by claim} \\ &= \mathrm{E}(X)\, e^{rz} \\ &\leq e\mathrm{E}(X) \end{aligned}$$

■

Lemma 2 *(version 1) Assume $X_1, \ldots, X_n$ are independent and on-off random variables such that $\forall i\ X_i \leq r^{-1}$ where $r = \frac{2\log(p^{-1}L)}{k}$. If $\sum_i B_r(X_i) \geq 4k$ then*

$$\Pr(\sum_i X_i > k) \geq \frac{1}{2}.$$

Proof. We will use Chernoff bound in the proof, which can be stated as follows: If $X_1, \ldots, X_n$ are on-off random variables such that $X_i \leq 1$, and $\sum_i \mathrm{E}(X_i) = M$, then

$$\Pr(|\sum_i X_i - M| > \delta M) \leq 2(\frac{e^\delta}{(1+\delta)^{(1+\delta)}})^m.$$

Using Lemma 1 and the assumption that $\sum_i B_r(X_i) \geq 4k$ we get that

$$\sum_i \mathrm{E}(X_i) \geq \frac{4}{e}k.$$

Therefore, choosing $\delta = \frac{4}{e} - 1$,

$$\Pr(\sum_i X_i \leq k) \leq 2(\frac{e^\delta}{(1+\delta)^{(1+\delta)}})^{\frac{4}{e}kr} < \frac{1}{2}.$$

■

Every item can be modeled by a set of on-off random variables, and the next lemma formalizes this.

Lemma 3 *For every random variable X there exists a set of independent on-off random variables $Y_1, Y_2, \ldots$ such that $\forall j$*

$$\Pr(2^j \leq X < 2^{j+1}) = \Pr(2^j \leq \sum_i Y_i < 2^{j+1}).$$

Proof. Random variable Y_i has the value 2^i with probability q_i. Define $p_i = \Pr(2^i \leq X < 2^{i+1})$.

Claim 2 $\max_j Y_j \leq \sum_j Y_j < 2\max_j Y_j$. *($Y_j$'s are powers of 2)*

$$\Pr(2^j \leq \sum_i Y_i < 2^{j+1}) = \Pr(\max_i Y_i = 2^j)$$

Therefore, we want, $\Pr(Y_i \text{ on and } Y_j \text{ off } \forall j > i) = p_i$, i.e.,

$$q_i \prod_{j>i}(1 - q_j) = p_i.$$

So, we can choose

$$q_i = \frac{p_i}{1 - \sum_{j>i} p_j}.$$

■

Lemma 4 *(version 2) Assume $X_1, \ldots, X_n$ are independent random variables such that $\forall i\ X_i \leq r^{-1}$ where $r = \frac{\log(p^{-1}L)}{2k}$. If there exists a constant c such that $\sum_i B_r(X_i) \geq ck$ then*

$$\Pr(\sum_i X_i > k) \geq \frac{1}{2}.$$

Proof. Model each X_i by a set $\{2Y_{i_1}, 2Y_{i_2}, \ldots\}$ using previous lemma.

$$X_i \le (\sum_j 2Y_{i_j}) \;\Rightarrow\; B_r(2Y_{i_j}) \ge ck$$

The variables Y_{i_j} are independent on-off items, so Lemma 2 applies. We use the lemma with $4k$ in place of k and assume $c \ge 16$, and we get that

$$\Pr(\sum_{i,j} 2Y_{i_j} > 4k) \ge 1/2.$$

Now notice that $X_i \ge \frac{1}{2}\sum_j Y_{i_j}$ for all i, therefore we get that

$$\Pr(\sum_i X_i > k) \ge 1/2.$$

■

Lemma 5 *(version 3) Assume $X_1, \ldots, X_n$ are independent random variables and $r = \frac{\log p^{-1}}{k}$. If there exists a constant c such that $\sum_i B_r(X_i) \ge ck$ then*

$$\Pr(\sum_i X_i > k) > p.$$

(This constant c turns out to be approximately 14).

Proof for this lemma is not given in the class, but you can refer to the paper via course homepage for a complete proof.

These lemmas are good for admission control on a network between pairs s, t using demands $B_r(X_{s,t})$. The good property about it is we can utilize c times the capacity, so we admit c times as much. The bad property is $B_r(X)$ is very sensitive to p.