Algorithmic aspects of communication networks	**Lecture 15**
OR 739 Spring 1997	**February 24, 1997**

Summary from the last class:
Let X be a random variable corresponding to the bandwidth a connection is going to require.
Define $B_r(X) = r^{-1}\log(E(e^{rX}))$ - the effective bandwidth
We had the following results:

1. $X_1, X_2, \quad , X_n$ are independent random variables implies:
   $B_r(\Sigma_i X_i) = \Sigma_i(B_r(X_i))$
2. If $B_r(X) \le C$ and $B \ge r^{-1}\log(p^{-1})$ then:
   $\Pr(X>B+C) \le p$
3. **Corollary** $X_1,\ldots, X_n$ independent, X is the sum of k independent copies of each X_i, $\Sigma_i B_r(X_i) \le C$, and $B \ge r^{-1}\log(p^{-1})$ implies:
   $\Pr(X > kC+B) \le p$

Statement of Jensen's Inequality which will be useful for proving the next lemma:
If the function u is convex, $E|X| < \infty$, and $E|u(X)| < \infty$ then $E(u(X)) \ge u(E(X))$.

Lemma If $X \ge 0$, and $r \ge 0$ then $E(X) \le B_r(X)$
Proof:
If we substitute in for $B_r(X)$ what we want to show is:

$$E(X) \le r^{-1}\log(E(e^{rX}))$$

Raise both sides to e^r

$$e^{rE(X)} \le E(e^{rX})$$

implies what we want to show is Jensen's inequality - so we are done. ♦

Goal: Use the effective bandwidth to get a routing and admission control algorithm for this problem.

We are given a network G = (V,E), $k \ge 0$ the capacity on each edge, L = maximum path length allowed, and the tolerated probability $p>0$ for overflow on a path.

s-t pairs of nodes arrive with associated random variables X_{st} corresponding to the required bandwidth.

Objective: accept as many pairs as possible in such a way that

$$\Pr(\exists\, e \in P_{st} \text{ that overflows}) \le p \qquad \forall \text{ s-t pairs}$$

where an edge e has overflow if

$$\Sigma_{st:e\in P_{st}} X_{st} > k$$

Other possible variants of this problem to consider would be to:
- guarantee that the probability some edge in the network overflows is small
  (too restrictive)
- guarantee that for every edge the probability that edge overflows is small
  (not restrictive enough)

Algorithm: Use the variant of routing and admission control with demand for s-t pairs dem(s,t) = $B_r(X_{st})$ and capacity k/2 on every edge. We will decide the value for r later on in our analysis.

Note: this version of routing and admission control is slightly different from the one we analyzed in class

assumption: dem(s,t)*$\log_2 L \leq k/2$ (capacity each edge = k/2)

Theorem (from admission control) If $r = 2\log(Lp^{-1})/k$ and the algorithm accepts $\mathcal{A}$ connections then the algorithm's routing is feasible but all routings that accept $\geq \mathcal{A}*O(\log(L\gamma))$ connections have an edge e with

$$\Sigma_{st:e\in P_{st}} \text{dem}(s,t) \geq 4k$$

where $\gamma = (\max_{st} \text{dem}(s,t))/(\min_{st} \text{dem}(s,t))$.

Lemma If $r = 2\log(Lp^{-1})/k$ then the resulting routing is feasible

Proof:

From our choose of routing algorithm we know that:

$$\Sigma_{st:e\in P_{st}} \text{dem}(s,t) \leq k/2 \quad \forall\, e \in E$$

by our definition of dem(s,t) this is the same as:

$$\Sigma_{st:e\in P_{st}} B_r(X_{st}) \leq k/2 \quad \forall\, e \in E$$

by the second result in the summary of last class this implies:

$$\Pr(\Sigma_{st:e\in P_{st}} X_{st} > k) \leq p/L$$

if C= B = k/2

(this requirement gives us our required condition for r:

$$B = k/2 \geq r^{-1}\log(Lp^{-1})$$

$$\Rightarrow \quad r = 2\log(Lp^{-1})/k \quad)$$

Since $\Pr(\Sigma_{st:e\in P_{st}} X_{st} > k) \leq p/L$ for all e in E, and the length of any path is most L we have:
The probability on any path that there exists an edge with overflow $\leq Lp/L = p$ ♦

Goal: Next we want to show that if many more pairs were accepted there would be a high probability of overflow.

From the theorem from admission control we know that if $\geq O(\log(L\gamma))$ times more pairs are accepted then there exists an edge e such that

$$\Sigma_{st:e\in P_{st}} B_r(X_{st}) \geq 4k$$

We would like a lemma similar to:

Lemma If $X_1, X_2, \ldots, X_n$ are independent random variables, $r = 2\log(Lp^{-1})/k$, and $\Sigma_i B_r(X_i) \geq 4k$ then

$$\Pr(\Sigma_i X_i > k) > p$$

Unfortunately, this lemma is not true.

If it were true we would have:

Theorem This algorithm is $O(\log(L\gamma))$ competitive.

We need an additional condition in order to make this lemma true which will discuss next class.