Algorithmic aspects of communication networks	**Lecture 14**
OR 739 Spring 1997	**February 21, 1997**

So far the only effect of routing in a network we considered was the change in the number of paths through each edge. In particular, this means that the bandwidth needed for every connection is the same. However, in more realistic models, some connections require more bandwidth than others.

Extending routing results to nonuniform bandwidth requirements

- Now each s-t connection pair has an associated bandwidth requirement $dem(s,t)$, that has to be available to establish the connection. Here to evaluate the load on each edge the function $f(e)$ is defined by

  $$f(e) = \sum_{e \in P_{s,t}} dem(s,t).$$

  Assuming that the amount of bandwidth requirement, $dem(s,t)$, is fixed over time, all the results we have studied can be extended to this model.

- Extensions can also be made to the model in which connections have varying demand (e.g. like in homework 1, # 3). Assuming that as soon as a connection is requested its demand as a function of time is immediately known, we can still extend the previous results. Now the bandwidth requirements are allowed to change over time. That is, as soon as connection s-t arrives, the amount of bandwidth requirement at each time τ, $dem(s,t,\tau)$, is known.

Random bandwidth requirements

- In a more realistic model, the arriving s-t connection request has a bandwidth requirement which is a random variable $X_{s,t}$.

- In addition, we will assume that for different pairs the $X_{s,t}$ variables are *independent*.

We will use the *effective bandwidth* approach. For each of the random bandwidth variables $X_{s,t}$, we are going to assign a positive real number $\mathrm{B}(X_{s,t})$ (*effective bandwidth* of the connection s-t). The idea is to use the number $\mathrm{B}(X_{s,t})$ in place of $dem(s,t)$, when routing.
For obtaining satisfactory algorithms, the function B has to be chosen carefully. Clearly if we set $\mathrm{B}(X) = \max X$, we would never overload any edge, but certainly this choice is very wasteful (intuitively, since the random variables $X_{s,t}$ are independent). Another possibility would be to use the expected value: $\mathrm{B}(X) = \mathrm{E}(X)$, but this not a *safe* choice.
The standard *safe* formula for effective bandwidth B is

$$\mathrm{B}_r(X) = r^{-1} \ln(\mathrm{E}(e^{rX})),$$

where r is a fixed positive number. Notice that $\mathrm{B}_r(X) \le \max X$ (clear) and also $\mathrm{E}(X) \le \mathrm{B}_r(X)$ (next time).

Why is it a good formula ?

First of all, note that if we do not use $\mathrm{B}(X) = \max X$, with positive probability a link might overflow. Hence, the best we can do is to guarantee a very low probability of overflow. In particular, we would like to prove a theorem of the form:

Theorem 1 *Let C be the capacity link and p the probability of allowed overflow. Assume that $X_1, \ldots, X_n$ are independent random variables and $\sum_i \mathrm{B}_r(X_i) \leq D$ (D to be determined). Then the probability of exceeding the link capacity, $\Pr(\sum X_i \geq C)$, is at most p.*

Notice that if $X_1, \ldots, X_n$ represent connections on the same link, then $X = \sum_i X_i$ is the total bandwidth needed.

First we need the following lemma.

Lemma 1 *If $X_1, \ldots, X_n$ are independent random variables, then*

$$\mathrm{B}_r(X) = \sum_i \mathrm{B}_r(X_i).$$

Proof:

$$\begin{aligned}
\mathrm{B}_r(X) &= r^{-1} \ln(\mathrm{E}(e^{r \sum_i X_i})) \\
&= r^{-1} \ln(\mathrm{E} \prod_i (e^{rX_i})) \\
&= r^{-1} \ln(\prod_i \mathrm{E}(e^{rX_i})) && \text{[because } X_1, \ldots, X_n \text{ are independent]} \\
&= r^{-1} \sum_i \ln(\mathrm{E}(e^{rX_i})) \\
&= \sum_i \mathrm{B}_r(X_i)
\end{aligned}$$

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We will also use Markov's inequality: for any nonnegative random variable X and $a > 0$,

$$\Pr(X \geq a) \leq \mathrm{E}(X)/a.$$

[To prove it, define

$$\bar{X} = \begin{cases} a & \text{if } X \geq a \\ 0 & \text{otherwise} \end{cases}$$

Clearly $X \geq \bar{X}$. Hence

$$\Pr(X \geq a) = \Pr(\bar{X} \geq a) = \frac{1}{a}\mathrm{E}(\bar{X}) \leq \frac{1}{a}\mathrm{E}(X). \quad]$$

Now coming back to theorem 1,

$$
\begin{aligned}
\Pr(X \geq C) &= \Pr(e^{rX} \geq e^{rC}) \\
&\leq e^{-rC}\mathrm{E}(e^{rX}) && \text{using Markov's ineq.} \\
&= e^{-rC}e^{\ln \mathrm{E}(e^{rX})} \\
&= e^{-rC}e^{r\mathrm{B}_r(X)} \\
&= e^{-r(C-\mathrm{B}_r(X))}
\end{aligned}
$$

Using the lemma, we would like that if $\mathrm{B}_r(X) = \sum_i \mathrm{B}_r(X_i) \leq D$ then $\Pr(X \geq C) \leq p$. By the previous calculation, we need D such that

$$
\begin{aligned}
e^{-r(C-D)} &\leq p \\
-r(C-D) &\leq \ln p \\
\ln \frac{1}{p} &\leq r(C-D) \\
D &\leq C - r^{-1}\ln \frac{1}{p}
\end{aligned}
$$

Hence, we have shown:

Theorem 1 *Let C be the capacity link and p the probability of allowed overflow. Assume that $X_1, \ldots, X_n$ are independent random variables, $\sum_i \mathrm{B}_r(X_i) \leq D$ and $D \leq C - r^{-1}\ln \frac{1}{p}$. Then the probability of exceeding the link capacity, $\Pr(\sum X_i \geq C)$, is at most p.*

We can also give the following buffer & time varying version. C is again the link capacity, but in addition there is a buffer with capacity b, which is used when the link capacity C overflows. Here we model a connection X_i lasting k units of time as the sum of k independent copies of X_i.

The proof of theorem 1 also gives the following.

Theorem 2 *Suppose that X is the sum of k independent copies of each of n random variables (in all, kn independent random variables). If $\sum_i \mathrm{B}_r(X_i) \leq C$ and $b \geq r^{-1}\ln \frac{1}{p}$, then the probability that X overflows, $\Pr(X \geq kC + b)$, is at most p.*

Admittedly, it is not realistic that consecutive time steps of one connection are independent. Other models studied in the literature mainly focus on the following:
(1) each X_i is an on-off source, which means that the random variable X_i takes either value 0 or a fixed positive value a_i (i.e. X_i is "on" when its value is a_i and "off" otherwise).
(2) it is assumed that the length of both the *on* and *off* periods are exponentially distributed (this assumption is usually called *exponentially holding times*).