Lecture 12 of February the 17^{th}

Off-Line Routing (Continued)

First let us remember the last lecture's notation and results:

c(e) := positive cost function.
f(e) := number of the paths on the edge e.
λ := max {f(e) : e is an arc}
$P_{s,t}$:= Path from s to t used in solutions.

From the last class, we have the following inequality and two Lemmas:

$$\sum_{s,t} dist_c(s,t) \le \sum_{s,t} \sum_{e \in P_{s,t}} c(e) = \sum_e c(e) f(e) \le \lambda \sum_e c(e)$$

Lemma 1: If $\lambda \sum_e c(e) \le (1+\delta) \sum_{s,t} dist_c(s,t)$, then $\lambda \le (1+\delta)\lambda^*$.

Lemma 2: If $c(e) = a^{f(e)}$ and $\log a \le \frac{1+\varepsilon}{\varepsilon\lambda} \log(\frac{m}{\varepsilon})$, then $\lambda \sum_e c(e) \le \frac{1+\varepsilon}{1-\varepsilon} \sum_e c(e) f(e)$.

In the routing Algorithm, a *phase* ends if λ decreases by a factor of 2. "*a*" is chosen via the following formula:

$\log a = 2\frac{1+\varepsilon}{\varepsilon\lambda_0} \log(\frac{m}{\varepsilon})$ where λ_0 is the value of λ at a start of a phase.

Corollary: "*a*" satisfies the condition of Lemma 2 throughout a phase.

Algorithm during a phase:
while *phase did not end*
for *all s,t pairs*
If $\sum_{e \in P_{s,t}} c(e) >> dist_c(s,t)$, *then reroute*

Note that costs are updated via $c(e) = a^{f(e)}$ whenever a pair is rerouted.

">>" stands for very large. We will next specify how large the sum of costs of all presently routed arcs need be for rerouting. Reroute if

$\sum_{e \in P_{s,t}} c(e) - dist_c(s,t) \ge \varepsilon \sum_{e \in P_{s,t}} c(e) + \frac{\varepsilon}{k} \lambda \sum_e c(e)$ where k is the number of (s,t) pairs.

Lemma: If during an iteration of the **while** loop no rerouting is done, then $\lambda \le \frac{1+\varepsilon}{1-3\varepsilon}\lambda^*$.

Proof: From Lemma 2

$$\lambda\sum_e c(e) \le \frac{1+\varepsilon}{1-\varepsilon}\sum_e c(e)f(e) = \frac{1+\varepsilon}{1-\varepsilon}\sum_{s,t}\sum_{e\in P_{s,t}} c(e)$$

From assumption of no rerouting

$$(1-\varepsilon)\sum_{e\in P_{s,t}} c(e) < dist_c(s,t) + \frac{\varepsilon}{k}\lambda\sum_e c(e)$$

Combining these two inequalities,

$$\lambda\sum_e c(e) \le \frac{1+\varepsilon}{(1-\varepsilon)^2}\sum_{s,t} dist_c(s,t) + \frac{1+\varepsilon}{(1-\varepsilon)^2}\varepsilon\lambda\sum_e c(e)$$

$$\lambda\sum_e c(e) \le \frac{1+\varepsilon}{1-3\varepsilon}\sum_{s,t} dist_c(s,t)$$

Now use Lemma 1 to get the desired result.

Let us define a measure ϕ for a given solution as $\phi = \sum_e c(e) = \sum_e a^{f(e)}$. Clearly, ϕ has a lower bound of m ($c(e) = 1$ *for all arcs if no path is routed*). Also initially, $\phi \le ma^{\lambda_0}$ (*because initially* $f(e) \le \lambda_0$ *for all arcs*). If ϕ can be shown to be decreasing, then we would know its range. The next lemma shows how fast ϕ decreases.

Lemma: If $\lambda \ge 8\varepsilon^{-2}\ln(m/\varepsilon)$ and we reroute, then $\phi_{new} \le \phi_{old}\left(1 - \frac{\log(m/\varepsilon)}{k}\right)$.

Proof: $\phi_{new} - \phi_{old} = \sum_e (a^{f_{old}(e)} - a^{f_{new}(e)})$. Suppose $P_{s,t}$ is the old path and $Q_{s,t}$ is the new one.
Define A := P\Q and B := Q\P.

$$\phi_{new} - \phi_{old} = \sum_{e\in A} c(e)(1-1/a) + \sum_{e\in \mathbf{B}} c(e)(1-a) \quad \text{where} \quad c(e) = a^{f_{old}(e)} .$$

"a" is determined by $\ln a \le 2\frac{1+\varepsilon}{\varepsilon\lambda}\ln(\frac{m}{\varepsilon}) \le \frac{\varepsilon}{4}(1+\varepsilon)$.

$$\phi_{new} - \phi_{old} = \left\{ \sum_{e \in A} c(e) - \sum_{e \in B} c(e) \right\} (1 - 1/a) + \sum_{e \in \mathbf{B}} c(e)(1 - a + 1 - 1/a)$$

Note that $a = e^{\ln a} \approx 1 + \ln a$. When $\ln a \le 1$, $1 + \ln a \le e^{\ln a} \le 1 + \ln a + (\ln a)^2$. However, we will use the lower bound as an approximation in what follows. The proof goes through with upper bound as well but more algebra is required.

[Proof will be completed on Wednesday.]