Admission Control: simple version

The **problem** we consider is as follows:

- Given a graph G with n nodes and m edges (undirected graph), and a bound $k \approx \log_2 m$.
- pairs of nodes arrive on-line
- When a pair (s,t) arrives we either reject the pair, or accept it, and route it in the graph so that on any edge we may route at most k paths.
- The objective is to maximize the number of paths accepted.

The **algorithm** is quite simple, and based on a cost function on the edges.

- For an edge e let $f(e)$ denote the number of paths routed through e.
- Define the cost of the edge e as $c(e) = 2^{f(e)}$.
- When a pair (s,t) arrives compute $cost(s,t)$, the minimum cost of a path in G from s to t (where the cost of a path is the sum of the cost of its edges).
- **Reject** the pair is the minimum cost of the path $cost(s,t) \geq 2^k$, **accept** it otherwise, and route accepted pairs on the minimum cost path.

Note that the algorithm did not explicitly make sure that no edge is used more than k times.

Lemma 0.1 *No edge is used by more than k paths.*

Proof. The proof is by contradiction. Let (s,t) be the $(k+1)$st pair that gets routed through edge e. Before this pair is routed edge e has $f(e) = k$ and hence cost $c(e) = 2^k$. The cost of the path through e contains the cost of edge e, and hence it is at least 2^k. Therefore if the minimum cost path is going through edge e, than the pair is rejected. ■

Next we want to argue that the solution is close to optimal. Consider a point in time, and $\mathcal{A}$ denote the set of pairs accepted by the algorithm, and $\mathcal{O}$ the set of pairs routed in an optimum solution among pairs considered so far. We want to prove the following theorem, proving that under the assumption that $k \approx log_2 m$ the algorithm is $\log m$-competitive.

Theorem 0.2 $|\mathcal{O}| \leq (2k+1)|\mathcal{A}|$.

We will prove the theorem through some lemmas. The idea is to compare both $|\mathcal{A}|$ and $|\mathcal{O}$ to $\sum_e c(e)$, the sum of the costs of all edges.

Lemma 0.3 $2^k|\mathcal{A}| \geq \sum_e c(e) - m$.

Proof. We prove the lemma by induction. Initially no pairs are accepted, so $\mathcal{A}$ is empty, and $c(e) = 2^0 = 1$ for all edges. So initially the two sides are both 0.

As the algorithm proceeds more pairs get accepted and the costs are increasing. Assume by induction that the inequality holds before a pair (s,t) is considered. If the pair is rejected, both sides of the inequality are unchanged, and hence the inequality continues to hold.

Assume now that the pair is accepted. Than the left hand side of the inequality increases by 2^k. Let P denote the minimum cots (s,t)-path. The cost of every edge on P doubles by definition, so the increase in the right hand side is the cost of path P, i.e., $cost(s,t)$. We prove that the inequality remains true, if we argue that the the increase of the left hand side is more than the increase of the right hand side, i.e., that $cost(s,t) \leq 2^k$. This is true, as the pair was accepted. ■

Next we will consider the optimum. Recall that c denotes the final cost function on the edges, and let $dist_c(s,t)$ denote the minimum cost of an (s,t)-path with cost c. Recall that $cost(s,t)$ is used to denote the minimum cost of an (s,t)-path when the pair (s,t) was considered, while $dist)_c(s,t)$ denotes the minimum cost with the final costs c. Observe also that the costs are monotone increasing during the algorithm, and therefore $cost(s,t) \leq dist_c(s,t)$ for all pairs (s,t).

Lemma 0.4 $\sum_{(s,t)\in\mathcal{O}} dist_c(s,t) \leq k\sum_e c(e)$.

Proof. Consider the paths $P_{s,t}$ used in the optimum solution. Clearly $dist_c(s,t)$ is at most the cost of the path $P_{s,t}$. We get the following chain of inequalities.

$$\sum_{(s,t)\in\mathcal{O}} dist_c(s,t) \leq \sum_{(s,t)\in\mathcal{O}} \sum_{e\in P_{s,t}} c(e) \leq k\sum_e c(e),$$

where the second inequality follows from switching the order of summation and using the fact that every edge participates in at most k of these paths. ■

Lemma 0.5 $2^k|\mathcal{O}\setminus\mathcal{A}| \leq k\sum_e c(e)$.

Proof. First recall, that $dist(s,t) \geq cost(s,t)$, and notice that all pairs in $\mathcal{O}\setminus\mathcal{A}$ have been rejected by the algorithm, hence have $cost(s,t) \geq 2^k$. So we have that $dist(s,t) \geq 2^k$.

We use the above lemma and this observation for the proof.

$$2^k|\mathcal{O}\setminus\mathcal{A}| \leq \sum_{(s,t)\in\mathcal{O}\setminus\mathcal{A}} dist(s,t) \leq k\sum_e c(e).$$

■

Now we can prove the main theorem.

Proof of the Theorem. From the lemmas we have the inequalities that

$$\begin{aligned} 2^k|\mathcal{A}| &\geq \sum_e c(e) - m \\ 2^k|\mathcal{O} \setminus \mathcal{A}| &\leq k\sum_e c(e). \end{aligned}$$

The assumption that $k \geq \log_2 m$ implies that $m \leq 2^k$. Notice further that the first pair presented to the algorithm is always accepted, as the initial cost of any path is at most n (initially all edges cost 1). Therefore $|\mathcal{A}| \geq 1$, and we have that $2^k|\mathcal{A}| \geq m$. Using this fact and the inequalities we get that

$$\begin{aligned} 2^k|\mathcal{O}| &\leq 2^k|\mathcal{O} \setminus \mathcal{A}| + 2^k|\mathcal{A}| \\ &\leq k\sum_e c(e) + 2^k|\mathcal{A}| \\ &\leq k(2^k|\mathcal{A}| + m) + 2^k|\mathcal{A}| \\ &\leq 2^k(2k+1)|\mathcal{A}|. \end{aligned}$$

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The idea of this proof is taken from a paper by Awerbuch, Azar and Plotkin. A more complete version of the statements and proof can be found in that paper: http://theory.stanford.edu/people/plotkin/rej.ps

A simple version similar this can be found in the survey of Serge Plotkin, also available on the Web: http://theory.stanford.edu/people/plotkin/jsac95.ps.