Speeding Up Belief Propagation for Early Vision

Daniel Huttenlocher

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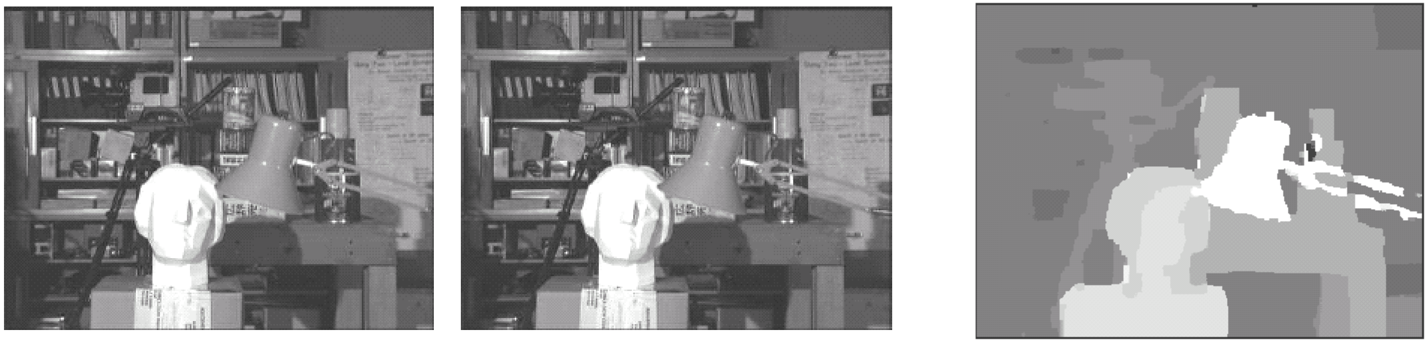
Joint work with Pedro Felzenszwalb

Overview

- Markov random field (MRF) models are broadly useful for low level vision
  - Framework for expressing tradeoff between spatial coherence and fidelity to data
- Substantial recent advances in algorithms for MRF models on grid graph
  - Two main approaches: graph cuts [BVZ01], loopy belief propagation (LBP) [WF01]
- Present three speedup techniques for LBP
  - Resulting methods hundreds of times faster than conventional techniques

Low Level Vision Problems

- Estimate label at each pixel
  - Stereo: disparity
  - Restoration: intensity
  - Segmentation: layers, regions
  - Optical flow: motion vector



Pixel Labeling Problem

- Find good assignment of labels to sites
  - Set $\mathcal{L}$ of k labels
  - Set $\mathcal{S}$ of n sites
  - Neighborhood system $\mathcal{N} \subseteq \mathcal{S} \times \mathcal{S}$ between sites
    - Consider case of (four connected) grid graph
- Undirected graphical model
  - Graph $\mathcal{G}=(\mathcal{S},\mathcal{N})$
  - Discrete random variable x_i over $\mathcal{L}$ at each site i
  - First order models
    - Maximal cliques in $\mathcal{G}$ of size 2

Form of Posterior

- Observations o
- Posterior distribution of labelings given observations

$$\Pr(x|o) \propto \Pr(o|x)\Pr(x)$$

- For first order model, prior factors as

$$\Pr(x) \propto \prod_{(i,j)\in\mathcal{N}} V(x_i,x_j)$$

- Further assume likelihood factors

$$\Pr(x|o) \propto \prod_{i\in\mathcal{S}} D_i(x_i) \prod_{(i,j)\in\mathcal{N}} V(x_i,x_j)$$

Estimation Problems

- Marginal probability at each node

$$\Pr(x_i | o)$$

- Maximize posterior (MAP)

$$\text{argmax}_x \prod_{i \in \mathcal{S}} D_i(x_i) \prod_{(i,j) \in \mathcal{N}} V(x_i, x_j)$$

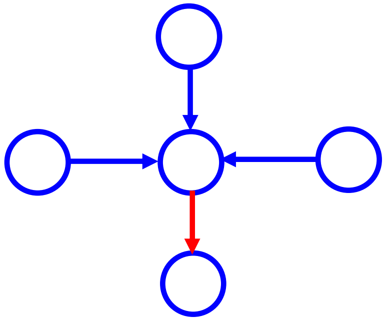
- Neither problem computationally tractable
  - NP hard for grid graph with 3 or more labels
- Various methods for approximate solution
  - Annealing, variational techniques, graph cuts using α-expansion, loopy belief propagation, …

Belief Propagation

- Iterative local update technique
  - Message passing, “nosy neighbor”
- Two forms
  - Sum product for estimating marginals
  - Max product for MAP estimation
- Exact solution when no loops in graph
- Update messages until “convergence” then compute distribution at each node
  - Sum product for marginals
  - Max product then max at each node for MAP

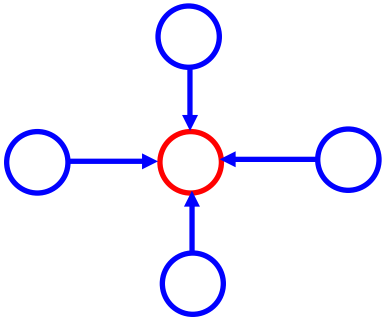
Sum Product

- At each step node j sends each neighbor a message, in parallel
  - Node j's view of i's labels



$$m_{j\to i}(x_i) = \sum_{x_j}\Big(D_j(x_j)\, V(x_j,x_i) \prod_{k\in\mathcal{N}(j)\setminus i} m_{k\to j}(x_j)\Big)$$

- After T iterations compute belief at each node
  - Using messages from neighbors and local data



$$b_j(x_j) = D_j(x_j) \prod_{i\in\mathcal{N}(j)} m_{i\to j}(x_j)$$

Max Product

- Min sum form with cost functions D′,V′ proportional to negative log potentials
- Message updates

$$m'_{j\to i}(x_i) = \min_{x_j}(D'_j(x_j) + V'(x_j,x_i) + \textstyle\sum_{k\in\mathcal{N}(j)\setminus i} m'_{k\to j}(x_j))$$

- After T iterations compute label minimizing value at each node

$$\operatorname{argmin}_{x_j} (D'_j(x_j) + \textstyle\sum_{i\in\mathcal{N}(j)} m'_{i\to j}(x_j))$$

  - Simple approach of separately minimizing at each node can be problematic

Three Techniques

- Memory requirements of BP large
  - Using bipartite form of graph can halve usage
- For vision problems $V(x_i,x_j)$ generally function of <u>difference</u> between labels
  - Enables computation of (discrete) messages in linear rather than quadratic time
- Number of iterations generally proportional to diameter of graph
  - Propagate information across grid
  - Using multi-grid methods can reduce to small constant number

Bipartite Graph ("Red-Black")

- Checkerboard pattern on grid defines a bipartite graph, $V=A \cup B$
- Alternating message updates of sets A,B yields messages $\overline{m}$ nearly same as m
  - Update messages from A on odd iterations and from B on even iterations
  - Then can show by induction when t odd (even)

$$\overline{m}^t_{i\to j} = \begin{cases} m^t_{i\to j} & \text{if i in A (i in B)} \\ m^{t-1}_{i\to j} & \text{otherwise} \end{cases}$$

  - Converges to same fixed point with half as many updates and half as much memory

Fast Message Updates

- Pairwise term V measuring label <u>difference</u>
- Sum product
  - Express as a convolution
  - O(klogk) algorithm using the FFT
  - Linear-time approximation algorithms for Gaussian models
- Min sum (max product)
  - Express as a min convolution
  - Linear time algorithms for common models using distance transforms and lower envelopes

Sum Product Message Passing

- When $V(x_i,x_j)=\rho(x_i-x_j)$ can write message update as convolution

  $$m_{j\to i}(x_i) = \sum_{x_j}(\rho(x_j-x_i)\ h(x_j))$$
  $$= \rho \star h$$

  - Where $h(x_j)= D_j(x_j) \prod_{k\in\mathcal{N}(j)\setminus i} m_{k\to j}(x_j))$
- Thus FFT can be used to compute in O(klogk) time for k values
  - Still somewhat large constants
- For ρ a (mixture of) Gaussian(s) do faster

Fast Gaussian Convolution

- A box filter has value 1 in some range

$$b_w(x) = \begin{cases} 1 \text{ if } 0 \le x \le w \\ 0 \text{ otherwise} \end{cases}$$

- A Gaussian can be approximated by repeated convolutions with a box filter
  - Application of central limit theorem, convolving pdf's tends to Gaussian
  - In practice, 4 convolutions [Wells, PAMI 86]

$$b_{w_1}(x) \star b_{w_2}(x) \star b_{w_3}(x) \star b_{w_4}(x) \approx G_\sigma(x)$$

  - Choose widths w_i such that $\sum_i (w_i^2 - 1)/12 \approx \sigma^2$

Convolution Using Box Sum

- Thus can approximate $G_\sigma(x) \star h(x)$ by cascade of box filters

$$b_{w_1}(x) \star (b_{w_2}(x) \star (b_{w_3}(x) \star (b_{w_4}(x) \star h(x))))$$

- Compute each $b_w(x) \star f(x)$ in time independent of box width w – sliding sum
  - Each successive shift of $b_w(x)$ w.r.t. $f(x)$ requires just one addition and one subtraction
- Overall computation just a few operations per label, O(k) with very low constant

Max Product Message Passing

- Can write message update as

$$m'_{j\to i}(x_i) = \min_{x_j}(\rho'(x_j - x_i) + h'(x_j))$$

  - Where $h'(x_j) = D'_j(x_j) \sum_{k\in\mathcal{N}(j)\backslash i} m'_{k\to j}(x_j))$
  - Formulation using minimization of costs, proportional to negative log probabilities
- Convolution-like operation over min,+ rather than $\sum,\times$ [FH00,FHK03]
  - No general fast algorithm like FFT
  - Certain important special cases in linear time

Commonly Used Pairwise Costs

- Potts model $\rho'(x) = \begin{cases} 0 \text{ if } x=0 \\ d \text{ otherwise} \end{cases}$
- Linear model $\rho'(x) = c|x|$
- Quadratic model $\rho'(x) = cx^2$
- Truncated models
  - Truncated linear $\rho'(x)=\min(d,c|x|)$
  - Truncated quadratic $\rho'(x)=\min(d,cx^2)$
- Min convolution can be computed in linear time for any of these cost functions

Potts Pairwise Model

- Substituting in to min convolution

  $$m'_{j\rightarrow i}(x_i) = \min_{x_j}(\rho'(x_j - x_i) + h'(x_j))$$

  can be written as

  $$m'_{j\rightarrow i}(x_i) = \min(h'(x_i), \min_{x_j} h'(x_j) + d)$$

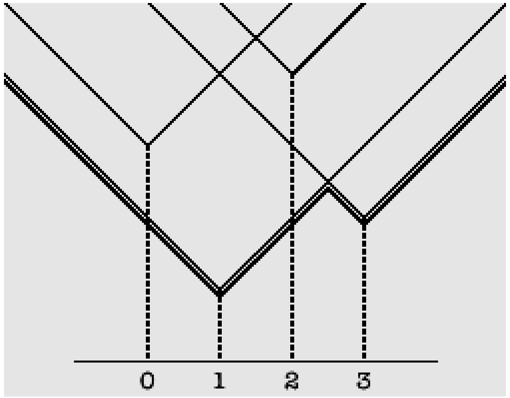
- No need to compare pairs x_i, x_j
  - Compute min over x_j once, then compare result with each x_i
- O(k) time for k labels
  - No special algorithm, just rewrite expression to obtain alternative (fast) computation

Linear Pairwise Model

- Substituting in to min convolution yields
  $$m'_{j \to i}(x_i) = \min_{x_j}(c|x_j - x_i| + h'(x_j))$$
- Similar form to the L_1 distance transform
  $$\min_{x_j}(|x_j - x_i| + 1(x_j))$$
  - Where $1(x) = \begin{cases} 0 & \text{when } x \in P \\ \infty & \text{otherwise} \end{cases}$
    is an indicator function for membership in P
- Distance transform measures L_1 distance to nearest point of P
  - Can think of computation as lower envelope of cones, one for each element of P

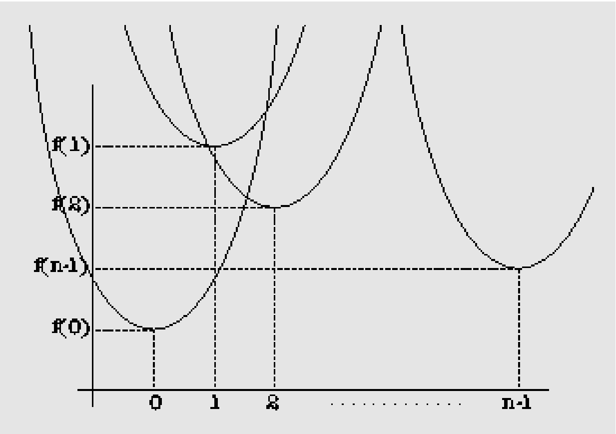
Using the L_1 Distance Transform

- Linear time algorithm
  - Traditionally used for indicator functions, but applies to any sampled function
- Forward pass
  - For x_j from 1 to k-1
    $m(x_j) \leftarrow \min(m(x_j), m(x_j\text{-}1)+c)$
- Backward pass
  - For x_j from k-2 to 0
    $m(x_j) \leftarrow \min(m(x_j), m(x_j+1)+c)$
- Example, c=1
  - (3,1,4,2) becomes (3,1,2,2) then (2,1,2,2)



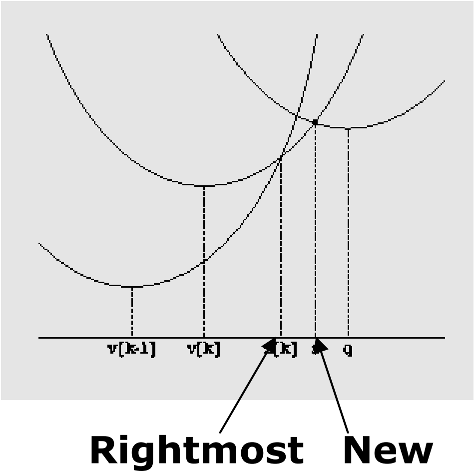
Quadratic Pairwise Model

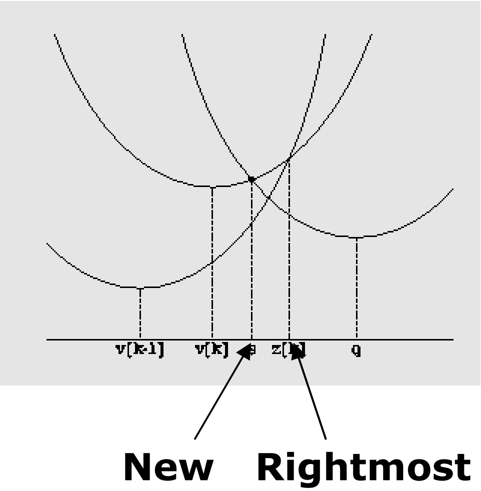
- Substituting in to min convolution yields
  $$m'_{j\rightarrow i}(x_i) = \min_{x_j}(c(x_j-x_i)^2 + h'(x_j))$$
- Again similar form to distance transform
- Compute lower envelope of parabolas
  - Each value of x_j defines a quadratic constraint, parabola rooted at $(x_j,h(x_j))$
  - In general can be done in O(klogk) [DG95]
  - Here parabolas are same shape and ordered, so O(k)



Lower Envelope of Parabolas

- Quadratics ordered $x_1 < x_2 < \ldots < x_n$
- At step j consider adding j-th one to LE
  - Maintain two ordered lists
    - Quadratics currently visible on LE
    - Intersections currently visible on LE
  - Compute intersection of j-th quadratic with rightmost visible on LE
    - If right of rightmost intersection add quadratic and intersection
    - If not, this quadratic hides at least rightmost quadratic, remove and try again





Running Time of Lower Envelope

- Consider adding each quadratic just once
  - Intersection and comparison constant time
  - Adding to lists constant time
  - Removing from lists constant time
    - But then need to try again
- Simple amortized analysis
  - Total number of removals O(k)
    - Each quadratic, once removed, never considered for removal again
- Thus overall running time O(k)

Code for Quadratic Pairwise Model

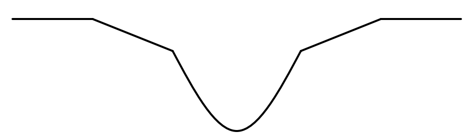
```
static float *dt(float *f, int n) {
  float *d = new float[n], *z = new float[n];
  int *v = new int[n], k = 0;
  v[0] = 0;
  z[0] = -INF; z[1] = +INF;
  for (int q = 1; q <= n-1; q++) {
    float s = ((f[q]+c*square(q)) (f[v[k]]+c*square(v[k])))
               /(2*c*q-2*c*v[k]);
    while (s <= z[k]) {
      k--;
      s  = ((f[q]+c*square(q))-(f[v[k]]+c*square(v[k])))
           /(2*c*q-2*c*v[k]);    }
    k++;
    v[k] = q;
    z[k] = s;
    z[k+1] = +INF; }
     k = 0;
  for (int q = 0; q <= n-1; q++) {
    while (z[k+1] < q)
      k++;
    d[q] = c*square(q-v[k]) + f[v[k]];  }
  return d;}
```

Combined Pairwise Models

- Truncated models
  - Compute un-truncated message m′
  - Truncate using Potts-like computation on m′ and original function h′

$$\min(m'(x_i), \min_{x_j} h'(x_j)+d)$$

- More general combinations
  - Min of any constant number of linear and quadratic functions, with or without truncation
    - E.g., multiple "segments"



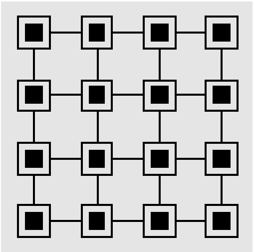
Fast Message Update Methods

- Efficient computation without assuming form of (discrete) distributions
  - Requires prior to be based on differences between labels rather than their identities
- Sum product
  - O(klogk) message updates for arbitrary discrete distributions over k labels using FFT
  - O(k) when pairwise clique potential a mixture of Gaussians using box sums
- Max product
  - O(k) for commonly used clique potentials

A Multi Grid Technique

- Number of message passing iterations T generally proportional to diameter of grid
  - Propagate information across the grid
- Use hierarchical approach to make independent of graph diameter
  - Previous work does this by changing the graph, building quad-tree with no loops [W02]
- Our approach is to define a hierarchy of problems with original graph structure
  - Initialize messages based on coarser levels

Hierarchy of Grids

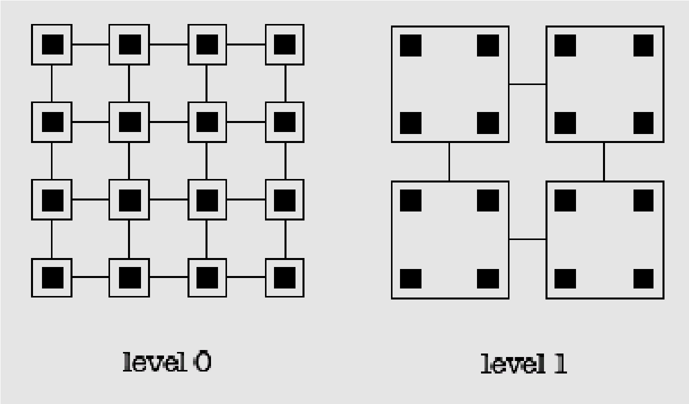


- Consider min sum case, rewrite minimization in terms of grid Γ

$$E(x) = \sum_{(i,j)\in\Gamma} D_{ij}(x_{i,j}) + \sum_{(i,j)\in\Gamma\backslash\mathcal{C}} V(x_{i,j}-x_{i+1,j}) + \sum_{(i,j)\in\Gamma\backslash\mathcal{R}} V(x_{i,j}-x_{i,j+1})$$

  - Where $\mathcal{C},\mathcal{R}$ last row and column of grid
- Can define family of grids Γ^0, Γ^1, …
  - An element of Γ^ℓ corresponds to $\varepsilon\times\varepsilon$ block of pixels, where $\varepsilon=2^\ell$
  - Labeling x^ℓ of Γ^ℓ assigns the pixels in each block a single label (from same set $\mathcal{L}$)

Problem Hierarchy



- Minimization problem at each level of the hierarchy

$$E^{\ell}(x^{\ell}) = \sum_{(i,j)\in\Gamma^{\ell}} D^{\ell}_{ij}(x^{\ell}_{i,j}) + \sum_{(i,j)\in\Gamma^{\ell}\backslash\mathcal{C}^{\ell}} V^{\ell}(x^{\ell}_{i,j}-x^{\ell}_{i+1,j}) + \sum_{(i,j)\in\Gamma^{\ell}\backslash\mathcal{R}^{\ell}} V^{\ell}(x^{\ell}_{i,j}-x^{\ell}_{i,j+1})$$

- Multi grid: final messages at one level as initial condition for next level, and so on
  - Small number of iterations if initial conditions close to final value

Hierarchical Data Term

- Finite element approach
- Assigning label α to block (i,j) at level ℓ equivalent to assigning α to each pixel in block

$$D^{\ell}_{ij}(\alpha) = \sum_{0 \leq u < \varepsilon} \sum_{0 \leq v < \varepsilon} D_{\varepsilon i+u, \varepsilon j+v}(\alpha)$$

  - Sum costs for all pixels in block
- Corresponds to product of probabilities, likelihood of observing pixels given label α
- Captures preference for multiple labels

Hierarchical Discontinuity Term

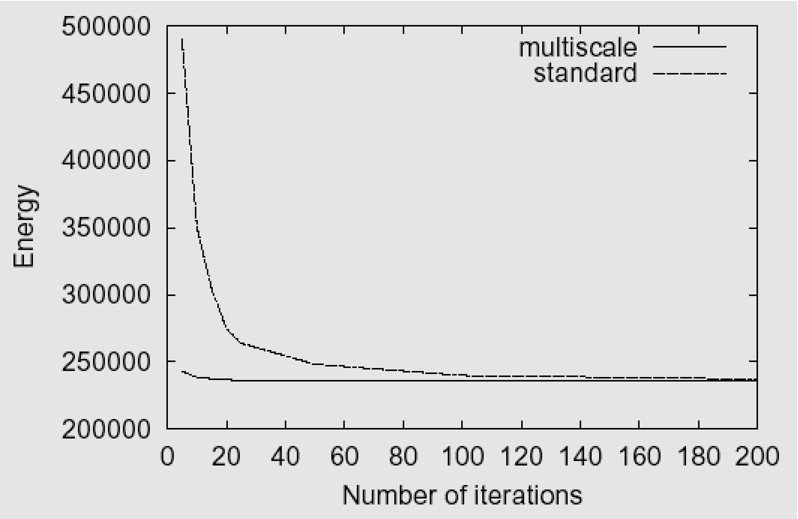
- Boundary between blocks length ε
  - Sum along boundary
- Separation between blocks ε
  - Finite difference, divide by separation

$$V^{\ell}(\alpha-\beta) = \varepsilon V\left(\frac{\alpha-\beta}{\varepsilon}\right)$$

- Produces different form depending on V
  - Linear, $V^{\ell}(x)=c|x|$
  - Quadratic, $V^{\ell}(x)=cx^2/\varepsilon$

Multi Grid Method

- Number of levels in hierarchy proportional to log image diameter
  - So propagation time small constant at top
- Same label set at each level
  - In contrast to pyramid methods
- In practice converges after a few iterations
  - Note each iteration just 1/3 more work than standard single level



Illustrative Results for Restoration

- Image restoration using MRF with truncated quadratic discontinuity cost
  - Not practical with conventional techniques, message updates 256^2
- Quadratic data term with no penalty for masked pixels
- Powerful formulation now practical
  - Largely abandoned except for small label sets

Gaussian noise and mask

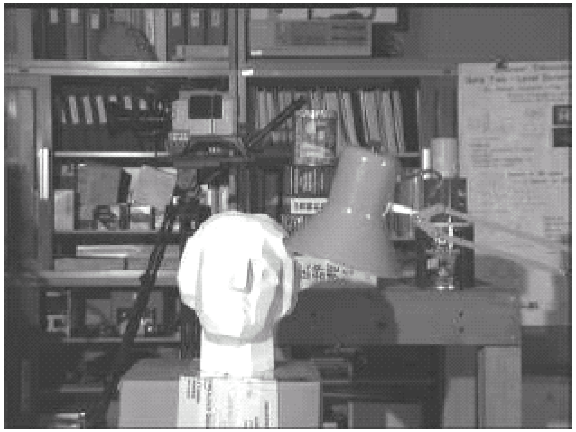
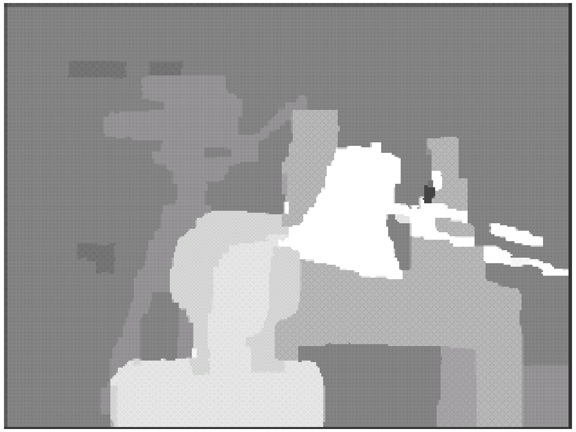
Illustrative Results for Stereo

- Truncated linear cost functions

$$D_i(x_i) = \min(d_b, |L(p_{i1}, p_{i2}) - R(p_{i1} - x_i, p_{i2})|)$$

$$V(x_i, x_j) = \min(d_s, |x_i - x_j|)$$

  - Runs in under a second for 30 disparity levels
- Same accuracy as slower methods
  - 12th in Middlebury benchmark (graph cuts 15th)

Extensions

- Fast message updates for max product in other cases
  - Discontinuity cost any convex function
    - Or truncated
  - Label set a multi-dimensional grid
    - E.g., flow vectors
  - Label sets not a regular grid
  - Possibly other "structured" label sets
- Additional labels such as occluded state for stereo can also be handled
  - Including penalty for length of occluded runs

Summary

- Fast methods for loopy belief propagation
  - Hundreds of times faster than previous methods
  - For discrete label space with potential functions based on differences between pairs of labels
  - Does not require parametric form of distributions
- Exact methods, not heuristic pruning or variational techniques
  - Except linear time Gaussian convolution which has (arbitrarily) small fixed approximation error
- Fast in practice, simple to implement
  - Code at http://people.cs.uchicago.edu/~pff/bp/