Workshop on Realizability `Preliminary Version`

Uniform provability realization of intuitionistic logic, modality and λ-terms

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Abstract

The intended meaning of intuitionistic logic is explained by the Brouwer-Heyting-Kolmogorov (*BHK*) provability semantics which informally defines intuitionistic truth as provability and specifies the intuitionistic connectives via operations on proofs. The problem of finding an adequate formalization of the provability semantics and establishing the completeness of the intuitionistic logic **Int** was first raised by Gödel in 1933. This question turned out to be a part of the more general problem of the intended realization for Gödel's modal logic of provability **S4** which was open since 1933. In this tutorial talk we present a provability realization of **Int** and **S4** that solves both of these problems. We describe the logic of explicit provability (**LP**) with the atoms "t is a proof of F" and establish that every theorem of **S4** admits a reading in **LP** as the statement about operations on proofs. Moreover, both **S4** and **Int** are shown to be complete with respect to this realization. In addition, **LP** subsumes the λ-calculus, modal λ-calculus, combinatory logic and provides a uniform provability realization of modality and λ-terms.

1 Introduction

In 1930 A. Heyting suggested the axiom system **Int** for intuitionistic logic ([8]) [3]. In 1931-34 Heyting and Kolmogorov made Brouwer's definition of intuitionistic truth explicit, though informal, by introducing what is now known as *Brouwer-Heyting-Kolmogorov (BHK) semantics* and is widely recognized as

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[3] The name **Int** will signify propositional intuitionistic logic.

the intended semantics of intuitionistic logic. *BHK* semantics gives an informal explanation of the truth of intuitionistic connectives. A statement is true if it has a proof, and a proof of a logically compound statement is given in terms of the proofs of its components. The description uses the unexplained primitive notions of *proof* and *construction*.

- a proof of a proposition $A \wedge B$ consists of a proof of A and a proof of B,
- a proof of $A \vee B$ is given by presenting either a proof of A or a proof of B,
- a proof of $A \rightarrow B$ is a construction which, given a proof of A returns a proof of B,
- absurdity $\perp$ is a proposition which has no proof and a proof of $\neg A$ is a construction which, given a proof of A, would return a proof of $\perp$.

The natural problem of finding an adequate formalization of the provability semantics and establishing the completeness of the intuitionistic logic **Int** with respect to this semantics remained open until recently despite a long history of studies in this area.

Note that the standard realizability semantics for **Int** is not faithful since there are realizable formulas that are not derivable in **Int**. First of all, following Kleene ([10]) we distinguish between intuitionistic and classical understanding of realizability semantics for intuitionistic theories. Intuitionistic realizability enjoys some nice completeness properties but does not provide an independent semantics for **Int**. For example, as follows from [21], a formula F is provable in intuitionistic predicate logic iff all arithmetical instances of F are provably realizable in a certain extension $\mathbf{HA}^+$ of intuitionistic arithmetic. Such a result relates **Int** with a formal theory based on the same **Int** and thus is not intended to give an independent semantics for the latter. On the other hand, classical realizabilities (Kleene realizability, function realizability, modified realizability, Medvedev's calculus of finite problems and its variants), give conditions necessary but not sufficient for **Int**.

Natural deduction proofs for **Int** can be transliterated by the Curry-Howard isomorphism into the language of typed λ-terms. The inductive definition of the Curry-Howard isomorphism goes along the lines of *BHK* clauses, where λ-terms play the role of *BHK* proofs. Though very important for establishing connections between derivations/formulas of **Int** and terms/types in λ-calculus, the Curry-Howard presentation does not give an independent semantical characterization for **Int**. Indeed, under this presentation the realization of a sentence is modulo to isomorphism a derivation of this sentence in the same **Int**. Loosely speaking, from the *BHK* semantics perspective, the Curry-Howard isomorphism provides a trivial solution: a formula F is true, by definition, if F is derivable in **Int**.

Despite strong similarities between Heyting's and Kolmogorov's descriptions of the provability semantics for **Int**, their approaches had fundamentally different objectives. Heyting explained propositional intuitionistic logic **Int** in terms of the intuitionistic understanding of constructions and proofs. His semantics gives a partial analysis of the intuitionistic meaning of a statement and does not intend to provide a foundation for **Int** independent of the intuitionistic assumptions. Kolmogorov in [11] intended to interpret **Int** on the basis of the usual mathematical notion of problem solution (e.g. proof), and thus to provide a *definition* of intuitionistic logic within classical mathematics.

For the purposes of formalization of *BHK* semantics it is important to distinguish between classical and intuitionistic interpretations of *BHK* clauses. We suggest the name *classical BHK semantics* for the former and *intuitionistic BHK semantics* for the latter. Thus, Kolmogorov's reading of **Int** as the logic of problem solutions may be considered classical *BHK* semantics.

We demonstrate that classical *BHK* semantics, in turn, admits an exact mathematical formalization, which indeed provides an adequate realization of **Int** on the basis of the usual classical notion of proof.

Gödel in 1933 introduced the modal logic of provability and explicitly defined **Int** in this logic. Gödel's provability logic has the modal axioms and rules $\Box F \to F$, $\Box(F \to G) \to (\Box F \to \Box G)$, $\Box F \to \Box\Box F$, $F \vdash \Box F$ (*necessitation* rule), admits all axioms and rules of classical logic, and therefore coincides with the classical modal logic **S4**. Gödel considered the translation $t(F)$ of an intuitionistic formula F into the classical modal language: "box each subformula of F". Gödel established that

$$\mathbf{Int} \vdash F \qquad \Rightarrow \qquad \mathbf{S4} \vdash t(F),$$

providing an exact reading of **Int** formulas as statements about provability in classical mathematics. He conjectured that the inverse $\Leftarrow$ also holds. This conjecture was eventually established in [16].

In one of his lectures [6] in 1938 (first published in 1995, see also [17]) Gödel sketched an explicit version of **S4** [4] with the basic proposition "t is a proof of F". Although this sketch does not contain exact definitions, it shows the way to explain the reflexivity principle for provability logic, which was the major difficulty in **S4**. Gödel's proposal generalized the problem of formalization of classical *BHK* semantics for **Int** to the problem of building an explicit provability logic: presumably, the former was derivable from the latter. The questions about an appropriate language and a complete set of axioms for explicit provability logic, as well as the question about its ability to realize **Int** and **S4** had remained open.

Kreisel in [14],[15] (apparently without knowledge of [6]) developed a formal theory of constructions with a basic predicate like Gödel's "t is a proof of F", but with only partial success. The system of conditions from [14],[15] even for the propositional language turned out to be inconsistent . Goodman in [7] suggested a way to fix this gap which, however led the resulting realizability away from the original *BHK* design. To the best of our knowledge,

[4] Gödel's sketch was rather clear about the propositional principles of explicit provability logic. It also mentioned possible principles involving the first order quantifiers, but was not specific on this matter. We consider the propositional part of Gödel's sketch only.

completeness of **Int** with respect to this Kreisel-Goodman realizability has not been established (cf. [17],[19],[22]).

2 About this talk

In this tutorial talk we present a recent solutions in [1], [2], [3] (cf. also a survey [9]) of the problems discussed above.

1. *We give the exact intended semantics and find a complete axiom system for the explicit provability logic sketched by Gödel in 1938.* We demonstrate that one more operation should be added to Gödel's sketch of the explicit provability logic (1938) in order to enable it to emulate the whole of **S4**. We call the resulting system the *Logic of Proofs* (**LP**).

2. *We find an adequate provability realization of the Gödel provability logic* **SL.** We establish that **LP** realizes all of **S4** by assigning proof terms to the modalities in every **S4**-derivation. This gives an adequate provability model for **S4** along the lines of Gödel's suggestion in [6].

3. *We formalize the classical BHK semantics for* **Int** *and establish the completeness of intuitionistic logic with respect to this realizability.* We consider two realizations of **Int** in **LP**. The first one is defined by Gödel's translation of intuitionistic formulas into modal language "box all subformulas", with the subsequent realization in **LP**. The second one is the McKinsey-Tarski translation ("box all atoms and implications") followed by the realization in **LP**. Each of those two semantics is established to be adequate for intuitionistic propositional logic. This confirms Kolmogorov's assumption of 1932 that intuitionistic logic **Int** coincides with the calculus of solutions to problems in classical mathematics. **LP** may be considered as the "unified logical technique dealing with two types of objects: statements and problems" meant by Kolmogorov in 1932 ([11],[12]). This also achieves the original objective of Gödel (1933) to define **Int** via the classical notion of proof.

4. *We demonstrate that* **LP** *subsumes modal λ-calculus thus providing a uniform provability realization of modality and λ-terms.* **LP** may be considered as a generalization of combinatory logic capable of iterating the type assignment ':'. In particular, **LP** can express the propositions of the form $t:(s:F)$, which are outside the scope of the usual combinatory logic. **LP** naturally contains the defined abstraction operator λ^*x which is an extension of the defined λ-abstraction operator λ^*x in combinatory logic (cf. [20]). This generalizes the Curry-Howard presentation of intuitionistic proofs as typed λ-terms. Moreover, through realizations in **LP** both modality and λ-terms receive a uniform provability semantics and thus may be treated as the objects of the same nature, namely proof polynomials.

3 Logic of Proofs

Definition 3.1 The language of Logic of Proofs (**LP**) contains

- the usual language of classical propositional logic
- proof variables $x_0, \ldots, x_n, \ldots$, proof constants $a_0, \ldots, a_n, \ldots$
- function symbols: monadic !, binary $\cdot$ and $+$
- operator symbol of the type "*term : formula*".

Terms are defined by the grammar

$$p ::= x_i \mid a_i \mid !p \mid p_1 \cdot p_2 \mid p_1 + p_2$$

We call these terms *proof polynomials* and denote them by $p,r,s,t\ldots$. Using t to stand for any term and S for any propositional letter, the formulas are defined by the grammar

$$\sigma ::= S \mid \sigma_1 \to \sigma_2 \mid \sigma_1 \wedge \sigma_2 \mid \sigma_1 \vee \sigma_2 \mid \neg\sigma \mid t{:}\sigma$$

The intended semantics for $p{:}F$ is "p is a proof of F" was made formal in [1] and [2].

Definition 3.2 The system **LP**. Axioms:

A0. Finite set of axiom schemes of classical propositional logic in the language of **LP**

A1. $t{:}F \to F$ *"verification"*

A2. $t{:}(F \to G) \to (s{:}F \to (t{\cdot}s){:}G)$ *"application"*

A3. $t{:}F \to\ !t{:}(t{:}F)$ *"proof checker"*

A4. $s{:}F \to (s{+}t){:}F, \quad t{:}F \to (s{+}t){:}F$ *"choice"*

Rules of inference:

R1. $$\frac{F \to G \qquad F}{G}$$ *"modus ponens".*

R2. $$\frac{}{c{:}\boldsymbol{A}},$$ *if* $\boldsymbol{A}$ *is an axiom A0 – A4, and c a proof constant* *"axiom necessitation".*

A *Constant Specification (CS)* is a finite set of formulas $c_1 : A_1, \ldots, c_n : A_n$ such that c_i is a constant, and A_i an axiom *A0 – A4*. Each derivation in **LP** naturally generates the *CS* consisting of all formulas introduced in this derivation by the *axiom necessitation* rule.

Remark 3.3 Proof constants in **LP** stand for proofs of "simple facts", namely propositional axioms and axioms *A1 – A4*. In a way the proof constants resemble atomic constant terms (*combinators*) of typed combinatory logic. A constant c_1 specified as $c_1 : (A \to (B \to A))$ can be identified with the combinator $\mathbf{k}^{A,B}$ of the type $A \to (B \to A)$. A constant c_2 such that $c_2{:}[(A \to$

$(B \to C)) \to ((A \to B) \to (A \to C))]$ corresponds to the combinator $\mathbf{s}^{A,B,C}$ of the type $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$. The proof variables may be regarded as term variables of combinatory logic, the operation "$\cdot$" as the application of terms. In general an **LP**-formula $t : F$ can be read as a combinatory term t of the type F. Typed combinatory logic $\mathbf{CL}_{\to}$ thus corresponds to a fragment of **LP** consisting only of formulas of the sort $t : F$ where t contains no operations other than "$\cdot$" and F is a formula built from the propositional letters by "$\to$" only.

There is no restriction on the choice of a constant c in *R2* within a given derivation. In particular, *R2* allows introducing a formula $c : A(c)$, or specifying a constant several times as a proof of different axioms from *A0* – *A4*.

Remark 3.4 The following constructive necessitation rule is admissible in **LP**:

$$\vdash F \quad \Rightarrow \ \vdash p : F \ \ \textit{for some proof polynomial } p$$

In fact this proof polynomial p is a formal blueprint of a derivation of F in **LP**.

Remark 3.5 The operations "$\cdot$" and "!" are present for single conclusion as well as on multiple conclusion proof systems. On the other hand, "+" is an operation for multiple conclusion proof systems only. Indeed, by *A4* we have $s : F \wedge t : G \to \ (s+t) : F \wedge (s+t) : G$, thus $s + t$ proves different formulas. The differences between single conclusion and multiple conclusion proof systems are mostly cosmetic. Usual proof systems (Hilbert or Gentzen style) may be considered as single conclusion if one assumes that a proof derives only the end formula (sequent) of a proof tree. On the other hand, the same systems may be regarded as multiple conclusion by assuming that a proof derives all formulas assigned to the nodes of the proof tree.

No single operator "$t :$" in **LP** is a normal modality since none of them satisfies the property $t : (P \to Q) \to (t : P \to t : Q)$. This makes **LP** essentially different from numerous polymodal logics, e.g. the dynamic logic of programs ([13]), where the modality is upgraded by some additional features. In turn, in the Logic of Proofs the modality is decomposed into a family of proof polynomials.

Theorem 3.6 ([1], [2]) **LP** *is sound and complete with respect to the natural provability semantics in a given consistent formal theory containing sufficient amount of arithmetic.* **LP** *is decidable and admits normalization of both Gentzen and natural style derivations.*

4 Realization of modal and intuitionistic logics

It is easy to see that the forgetful projection of **LP** is correct with respect to **S4**. Let F^o be the result of substituting $\Box X$ for all occurrences of $t : X$ in F, and $\Gamma^o = \{F^o \mid F \in \Gamma\}$ for any set Γ of **LP**-formulas.

Lemma 4.1 *If* **LP** $\vdash F$, then ***S4*** $\vdash F^o$.

In fact, the converse also holds: **LP** suffices to realize any theorem of **S4**.

Definition 4.2 By an **LP**-*realization* of a modal formula F we mean an assignment of proof polynomials to all occurrences of the modality in F. Let F^r be the image of F under a realization r. A realization r is *normal* if all negative occurrences of $\Box$ are realized by proof variables.

Theorem 4.3 *If* **S4** $\vdash F$, *then* **LP** $\vdash F^r$ *for some normal realization* r.

Example 4.4 **S4** $\vdash (\Box A \wedge \Box B) \to \Box(A \wedge B)$

In **LP** the corresponding derivation is

1. $c{:}(A \to (B \to A \wedge B))$, by *R2*,
2. $x{:}A \to (c \cdot x){:}(B \to A \wedge B)$, from 1, by *A2*,
3. $x{:}A \to (y{:}B \to (c \cdot x \cdot y){:}(A \wedge B))$, from 2, by *A2* and propositional logic,
4. $x{:}A \wedge y{:}B \to (c \cdot x \cdot y){:}(A \wedge B))$, from 3, by propositional logic.

Example 4.5 **S4** $\vdash (\Box A \vee \Box B) \to \Box(A \vee B)$.

In **LP** the corresponding derivation is

1. $a{:}(A \to A \vee B)$, $\quad b{:}(B \to A \vee B)$, by *R2*,
2. $x{:}A \to (a{\cdot}x){:}(A \vee B)$, $\quad y{:}B \to (b{\cdot}y){:}(A \vee B)$, from 1, by *A2*,
3. $(ax){:}(A \vee B) \to (ax+by){:}(A \vee B)$, $\quad (by){:}(A \vee B) \to (ax+by){:}(A \vee B)$, by *A4*,
4. $(x{:}A \vee y{:}B) \to (ax+by){:}(A \vee B)$, from 3, by propositional logic.

Remark 4.6 **S4** is nothing but a lazy version of **LP** that does not distinguish between the proof polynomials. Each theorem of **S4** admits a decoding via **LP** as a statement about specific proofs. The language of **LP** is more rich than that of **S4**. In particular, **S4** theorems admit essentially different realizations in **LP**. For example, consider two theorems of **LP** having the same modal projection:

$$x{:}F \vee y{:}F \to (x+y){:}F \quad \text{and} \quad x{:}F \vee x{:}F \to x{:}F.$$

The former of these formulas is a meaningful specification of the operation "+". In a contrast, the latter one is a trivial tautology.

A recipe for using **S4** as a provability logic could be the following: find a proof of F in **S4** and then run a realization algorithm to recover the provability meaning of F.

Definition 4.7 Let $gk(F)$ denote a translation of an intuitionistic formula F into the plain modal language that puts the prefix $\Box$ in front of all subformulas in F (*Gödel-Kolmogorov translation*). Under $mt(F)$ we understand the translation that prefixes with $\Box$ only atoms and implications in F (*McKinsey-Tarski translation*). A propositional formula F is *GK-realizable* (*MT-realizable*) if there exists a normal realization r such that $gk(F)^r$ ($mt(F)^r$) is derivable in **LP**.

Theorem 4.8 (Realization of intuitionistic logic) *For any* **Int**-*formula* F

1. $\mathbf{Int} \vdash F \quad \Leftrightarrow \quad F$ *is GK-realizable,*
2. $\mathbf{Int} \vdash F \quad \Leftrightarrow \quad F$ *is MT-realizable*

Note that *GK*-realizability may be regarded as a formalization of the Kolmogorov calculus of problems from [11] by reading "problem solutions" as "proofs". This realizability gives a plausible formalization of Kolmogorov's calculus of problems [11]. Propositional atoms are interpreted as *atomic problems*, namely statements of the sort $t:S$ meaning "*t is a proof of S*". Intuitionistic connectives are given precise meaning according to [11].

Example 4.9 Let S, T be propositional letters. Consider the formula

$$F \equiv (\neg S \vee T) \rightarrow (S \rightarrow T),$$

obviously provable in **Int**. The corresponding translations of this formula to the modal language are (in both cases the outermost $\Box$'s are suppressed for briefly):

$$mt(F) = (\Box\neg\Box S \vee \Box T) \rightarrow \Box(\Box S \rightarrow \Box T),$$
$$gk(F) = \Box(\Box\neg\Box S \vee \Box T) \rightarrow \Box(\Box S \rightarrow \Box T).$$

We will present normal realizations in **LP** for each of $mt(F)$ and $gk(F)$.

The following is a derivation in **LP** with a simultaneous construction of a normal realization of $mt(F)$.

1. $\neg x:S \rightarrow (x:S \rightarrow y:T)$, by classical logic;
2. $a:[\neg x:S \rightarrow (x:S \rightarrow y:T)]$, by necessitation rule 3.4. Note that here a is a product of some axiom constants with obvious specifications;
3. $z:(\neg x:S) \rightarrow (a\cdot z):(x:S \rightarrow y:T)$, from 2, by *A2*;
4. $y:T \rightarrow (x:S \rightarrow y:T)$, axiom of propositional logic *A0*;
5. $b:[y:T \rightarrow (x:S \rightarrow y:T)]$, from 4, by axiom necessitation *R2*;
6. $!y:y:T \rightarrow (b\cdot !y):(x:S \rightarrow y:T)$, from 5, by *A2*;
7. $y:T \rightarrow !y:y:T$, axiom *A3*;
8. $y:T \rightarrow (b\cdot !y):(x:S \rightarrow y:T)$, from 6, 7, by classical logic;
9. $(z:(\neg x:S) \vee y:T) \rightarrow (a\cdot z + b\cdot !y):(x:S \rightarrow y:T)$, from 3, 8, by *A4*.

In the case of $gk(F)$ the realization is constructed along the following derivation in **LP**.

1. $\neg x:S \rightarrow (x:S \rightarrow y:T)$, by classical logic;
2. $z:(\neg x:S) \rightarrow \neg x:S$, axiom *A1*;
3. $z:(\neg x:S) \rightarrow (x:S \rightarrow y:T)$, from 1, 2;
4. $y:T \rightarrow (x:S \rightarrow y:T)$, axiom of propositional logic *A0*;
5. $(z:(\neg x:S) \vee y:T) \rightarrow (x:S \rightarrow y:T)$, from 3, 4, by classical logic;

6. $c\!:\!H$, when H is from 5, by necessitation rule 3.4. Here c is a ground proof polynomial, easily recoverable from the derivation of 5.

7. $u\!:\!(z\!:\!(\neg x\!:\!S)\vee y\!:\!T) \rightarrow (c\cdot u)\!:\!(x\!:\!S\!\rightarrow\! y\!:\!T)$, from 6, by *A2*.

5 Realization of modal λ-calculus

Roughly speaking, **LP** is an advanced system of combinatory logic that accommodates not only the "application" operation, but also "proof checker" and "choice". These operations subsume the simply typed λ-calculus together with the modal logic **S4**, and thus the entire modal λ-calculus ([4], [18]). In particular, **LP** creates an environment where modality and λ-terms are objects of the same nature, namely proof polynomials. Another way to look at it: modal logic is a forgetful projection of a combinatory logic with dependent types enriched by the operations "proof checker" and "choice".

Such uniform understanding of modalities and λ-terms as proof polynomials is more flexible than the usual modal λ-calculus. In particular, there is no modal λ-term $t(x)$ such that $x : A \rightarrow\ t(x) : \Box A$ whereas there is a proof polynomial $!x$ ("proof checker") in **LP** such that $x\!:\!A\!\rightarrow\, !x\!:\!x\!:\!A$ holds.

The realization of **S4** in **LP** provides a fresh look at modal logic and its applications in general. Proof polynomials reveal the dynamic character of modality. It raises the general question of finding explicit counterparts to all major modal logics. Such areas as modal λ-calculi, λ-calculi with types depending on terms, non-deterministic λ-calculi, etc., could benefit from viewing their semantics as proof polynomials delivered by **LP**.

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