Problem Set I Solution

Spring 1998
CS522

1. Suppose you exercise the American put option at $t = 0$. To exercise a put option, you have to deliver a stock. Therefore, you gain nothing after you finance to buy a stock since the stock price and the exercise price are the same at $t = 0$. On the other hand, if you exercise the option at any $t > 0$, you will get $K - S(t)$ which is positive. It's not optimal to exercise at $t = 0$.

2. Suppose you exercise the option at time t and invest the payoff in the money market account, if any. You end up with $max(K - S(t), 0)e^{r(T-t)}$ at the maturity T. On the other hand, the payoff will be K when you exercise the option at the maturity. If stock price at t is low enough and the interest rate r is high, then there is a chance that the payoff from the early exercise exceeds K. So, it is not necessarily optimal to exercise the option at T.

3. After solving the linear equation

$$\begin{aligned} SUm + RB &= SU - K \\ SUm + RB &= 0 \end{aligned} \qquad (0.1)$$

we have $m = \dfrac{SU - K}{S(U - D)}$ and $B = \dfrac{D(K - SU)}{R(U - D)}$

Substitute this into $c(0) = mS + B$

$$c(0) = \left(\frac{R-D}{U-D}\right)\left(\frac{SU-K}{R}\right) = \pi\left(\frac{SU-K}{R}\right) = \frac{\pi(SU-K) + (1-\pi)0}{R} \text{ where } \pi = \frac{R-D}{U-D}$$

4. From the recursive formula in the lecture note,

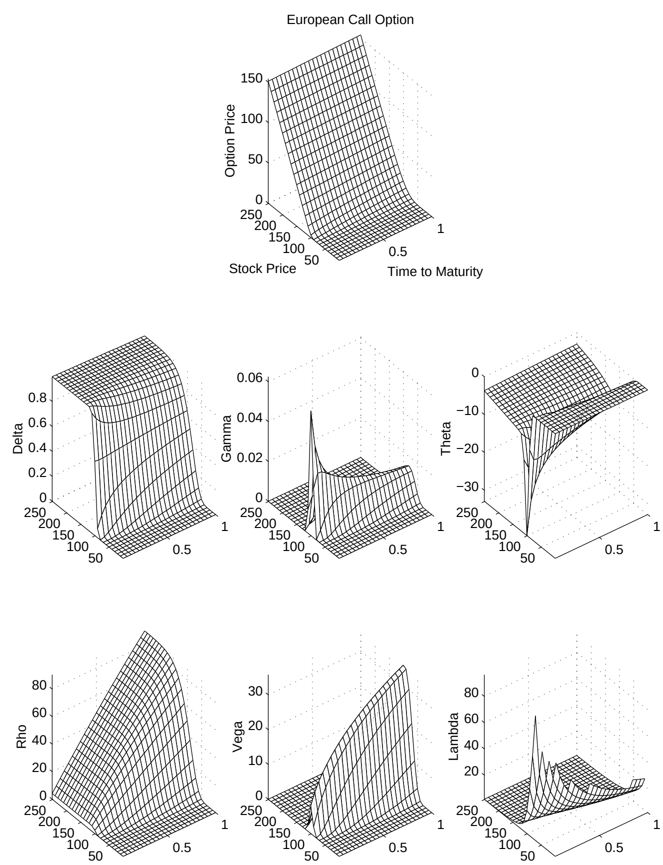
$$\begin{aligned} c(1{:}u) &= \frac{[\pi max(SU^2 - K, 0) + (1-\pi)max(SUD - K, 0)]}{R} \\ c(1{:}d) &= \frac{[\pi max(SUD - K, 0) + (1-\pi)max(SD^2 - K, 0)]}{R} \end{aligned} \qquad (0.2)$$

Therefore,

$$c(0) = \frac{[\pi c(1{:}u) + (1-\pi)c(1{:}d)]}{R} = \frac{\sum_{j=0}^{2}\binom{2}{j}\pi^j(1-\pi)^{2-j}max(SU^jD^{2-j} - K, 0)}{R^2} \qquad (0.3)$$

5. (a) The following is the code which generates the graphs.

```
S = 1:10:251; K = 100; R = .1; T = [.5:.5:12]/12; SIG = .3; Q = .025;
[Smat, Tmat] = meshgrid(S, T);

call = blsprice(Smat, K, R, Tmat, SIG, Q);
subplot(3,3,2); mesh(Tmat, Smat, call);
xlabel('Time to Maturity'); ylabel('Stock Price'); zlabel('Option Price');
title('European Call Option')

delta = blsdelta(Smat, K, R, Tmat, SIG, Q);
subplot(3,3,4); mesh(Tmat, Smat, delta); zlabel('Delta');

gamma = blsgamma(Smat, K, R, Tmat, SIG, Q);
subplot(3,3,5); mesh(Tmat, Smat, gamma); zlabel('Gamma');

theta = blstheta(Smat, K, R, Tmat, SIG, Q);
subplot(3,3,6); mesh(Tmat, Smat, theta); zlabel('Theta');

rho = blsrho(Smat, K, R, Tmat, SIG, Q);
subplot(3,3,7); mesh(Tmat, Smat, rho); zlabel('Rho');

vega = blsvega(Smat, K, R, Tmat, SIG, Q);
subplot(3,3,8); mesh(Tmat, Smat, vega); zlabel('Vega');

lambda = blslambda(Smat, K, R, Tmat, SIG, Q);
subplot(3,3,9); mesh(Tmat, Smat, lambda); zlabel('Lambda');
```



Financial interpretation

Option price : The price of a call option is increasing as the stock price goes up and decreasing as it gets to the maturity since the uncertainty gets smaller as time approaches the maturity.

Delta : Delta of a derivative is the rate of change of its price relative to the price of the underlying asset. First, Delta is positive since option price is increasing as the stock price goes up. Delta for $S \ll K$ is almost zero since the option price is almost zero and its movement is negligible with respect to the stock price. For $S \gg K$, it's almost sure that the option will be exercised and the change of the option price is almost the same as the change of the stock price since the payoff is close to $S - K$, which explains why delta is almost 1 in that case.

Gamma : Gamma is the rate of change of delta relative to the price of underlying asset. Gamma is almost zero for $S \ll K$ and S $\gg K$ since delta is constant in that case. Gamma is the biggest around the exercise price since there is a huge change in delta. Gamma has a peak at the maturity since delta changes dramatically at the maturity. (Actually Gamma is infinity at $(S, t) = (K, 0)$ since delta has a jump at the point.)

Theta : Theta is the rate of change in the price of a derivative security relative to time. Note that theta is negative since the option price decreases as it gets to the maturity. When the stock price is much smaller than the exercise price, the change in the option price is negligible, which explains the small magnitude of theta. When the stock price is close to the exercise price, the option price undergoes a huge change from some positive value to almost zero, specially at the exercise price and it explains the peak at K.

Rho : Rho is the rate of change in the option price relative to the underlying security's risk-free interest rate. Rho is an increasing function with respect to the time to maturity since the interest rate has smaller influence when there is little time left before the maturity. Rho is close to zero for $S \ll K$ since the option price is almost zero.

Vega : Vega is the rate of change in the price of a derivative security relative to the volatility of the underlying security. Vega is positive since the payoff of an option increases as the stock price fluctuates more. Vega is an increasing function of the time to maturity since we have more chance to make profit when there is more time left. Volatility has no effect when $S \ll K$ or $S \gg K$ since it's almost certain whether we exercise the option or not. For the stock price close to the exercise price, the option price changes dramatically as volatility changes explaining the peak around the exercise price.

Lambda : Lambda, also known as the elasticity of an option, represents the percentage change in the price of an option relative to a 1% change in the price of the underlying security. Lambda is positive since it has the same sign as delta of the option. Lambda is huge for $S \ll K$ near the maturity since the option price is very small and has small magnitude for $S \gg K$ since the option price is high.

(b)
```
I = 12:13; J = 8:12;
  vol = blsimpv(Smat(I, J), K, R-Q, Tmat(I, J), call(I, J))
vol =
    0.3231    0.2985    0.2973    0.2950    0.2903
    0.3197    0.2983    0.2970    0.2945    0.2897
norm(vol(:) - .3)
```

```
ans =
    0.0346
```

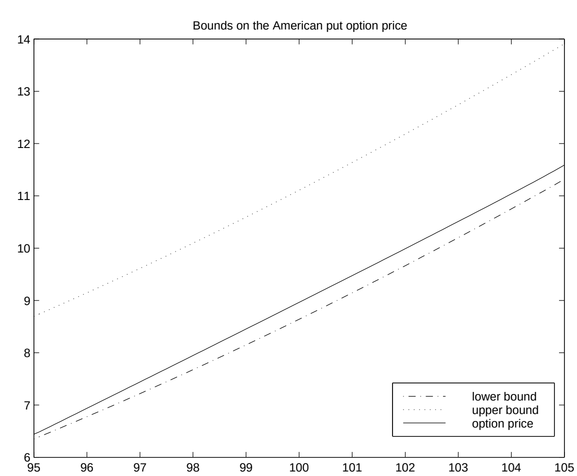
The implied volatilities are close to the given volatility as we expected. The interest rate is adjusted since `blsimpv` doesn't take the dividend rate into account.

6. Consider two portfolio :
Portfolio A consists of one European call option and the cash of $D + Ke^{-rT}$.
Portfolio B consists of one European put and a stock.
At the maturity, both portfolios have the same value, $max(S, K) + dividend$, showing the put-call parity holds.

7. (a) From the put-call parity for European options, $c + Ke^{-rT} = p + S$
Applying $c = C$ and $p < P$, we have $C + Ke^{-rT} < P + S$.

(b) Let's consider the portfolios from the hint.
Suppose the put option is exercised at t (assuming $S(t) < K$). Portfolio A has the value $c(t) + Ke^{rt}$ and portfolio B becomes K at t
Suppose the put option is not exercised before the maturity. At T, portfolio A is $max(S, K) + Ke^{rT} - K$ and portfolio B is $max(S, K)$.
In any case, Portfolio A ends up with more value, showing $c + K > P + S$. Since $c = C$, $C + K > P + S$.

8.
```
S = 100; R = .1; T = .25; SIG = .5; D = 0; K = 95:.1:105; dT = .01; FLAG = 0;
 i = 0;
 for k = K
   i = i+1;
  [pr,opt] = binprice(S, k, R, T, dT, SIG, FLAG, D); put(i) = opt(1,1); end
 call = blsprice(S, K, R, T, SIG);
plot(K, call - S + K*exp(-R*T), '-.', K, call - S + K, ':', K, put, '-');
 legend('lower bound', 'upper bound', 'option price');
 xlabel('Exercise Price : K');
 title('Bounds on the American put option price');
```



9.

x_i : number of options **(0.4)**
Δ_i : deltas
γ_i : gammas
VE_i : vegas
v_i : option values $i = 1, 2, 3, 4$

(a)

$$min \left(\sum_{i=1}^{4} x_i \Delta_i\right)^2 + \left(\sum_{i=1}^{4} x_i \gamma_i\right)^2 + \left(\sum_{i=1}^{4} x_i VE_i\right)^2 \quad \textbf{(0.5)}$$

$$\text{s. t.} \quad \sum x_i v_i = 17,000$$

$$x \geq 0$$

You can take the sum of 1-norm as an objective function instead of sum of squares.

(b) Now we have to solve the system of equations

$$\sum x_i \Delta_i = 0 \quad \textbf{(0.6)}$$
$$\sum x_i v_i = 17,000$$

This system is underdetermined and has infinite number of solutions.

When shorting is not allowed, the problem becomes to find a point in the polyhedron $\left\{\sum x_i \Delta_i = 0, \sum x_i v_i = 17000, x \geq 0\right\}$.

(c)

$$max \quad \sum x_i v_i r_i$$

$$\text{s. t.} \quad \sum x_i \Delta_i = 0 \quad \textbf{(0.7)}$$
$$\sum x_i v_i = 17000$$
$$x \geq 0$$