CS – 522 Computational Tools and Methods in Finance
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Lecture 6: Portfolio Theory

1. Preferences

a. Choices

Let X be a discrete set of outcomes (in dollars), e.g. X = [0, 1, 2, ..., 10,000].

Let P be a *finite* set of probability distributions over X.

$$p \in P \text{ implies that } p(x) \geq 0 \text{ and } \sum_{x \in X} p(x) = 1.$$

Let P include all "point" distributions, i.e. p(x) = 1 for some $x \in X$.

Remark: finiteness of X and P is for simplicity.

b. Investors

Investors are represented by their preferences (a preference relation) on P.

A binary relation on P is a collection of pairs

u = {(p,q) satisfying some property for p,q ∈ P }.

We say p **u** q if (p,q)∈ **u**. We say that p **u(s)** q if p **u** q and not q **u** p.

Remark. **u** is "prefers or is indifferent", **u(s)** is "strictly prefers"

A preference relation is a binary relation u on P such that it is:

1. transitive, i.e. p **u** q and q **u** r implies p **u** r
2. complete, i.e. given p and q, either p **u** q or q **u** p.

Lemma (Existence of a Utility Function):

Given **u** is a preference relation. There exists a U: P→ R (the real line) such that

U(p) ≥ U(q) if and only if p **u** q .

Proof: skip.

U is called a utility function.

This is not enough for analysis, look for an expected utility representation of **u**.

Lemma (Existence of Expected Utility Representation):

Given **u** is a preference relation and

1. Independence Axiom
   For all p,q,r $\in$ P and a $\in$ (0,1], p **u(s)** q implies [ap+(1-a)r] **u(s)** [aq+(1-a)r].

2. Archimedean Axiom
   For all p,q,r $\in$ P, if p **u(s)** q **u(s)** r then there exists a, b $\in$ (0,1) such that
   [ap + (1-a)r] **u(s)** q **u(s)** [bp + (1-b)r].

Then, there exists a V: X $\rightarrow$ R such that

$$U(p) = \sum_{x \in X} V(x)p(x) = E_p(V(x)) .$$

Proof: skip.

So, from here on, investor's preferences have an expected utility representation.

Remark. Called rational preferences.

c. Prefers More Wealth to Less

Usually assume that V(x) > V(y) for x > y.

More wealth is preferred to less.

d. Risk Aversion

An investor is said to be risk averse if

Given w,x,y $\in$ R, if ax + (1-a)y = w and 0<a<1 , then V(w) > a V(x) + (1-a) V(y).

Remarks.

1. Risk averse investors prefer the sure wealth w to the gamble with an expectation of w.

3. Risk neutral if indifferent. Risk loving if dislikes.

A function V is strictly concave if

x,y $\in$ R and 0<a<1, V(ax+(1-a)y) > aV(x) + (1-a)V(y)

Lemma (Strict Concavity).

An investor is risk averse if and only if V is strictly concave.

Remark. Usually only study risk averse investors.

e. Differentiability

Usually assume that V is twice continuously differentiable. Given that investors prefer more wealth to less and that they are risk averse, we have that:

$V'(x) > 0$ and $V''(x) < 0$.

3. Portfolio Theory Problem

a. General Formulation

Consider an investor with probability beliefs represented by $p \in P$ and a utility function $V: R \to R$.

Let there be one period in time, from 0 to 1.

Consider a collection of random variables:

$r_1, r_2, \cdots, r_N$. These random variables represent the returns on risky assets at time 1.

A time 0 portfolio is a collection of weights $(w_1, \cdots, w_N)$ such that $\sum_{i=1}^{N} w_i = 1$.
The time 1 wealth of a portfolio is

$W = \sum_{i=1}^{N} w_i r_i$. This is a random variable.

The investor's portfolio problem is:

to choose a portfolio $(w_1, \cdots, w_N)$ such that

$E(V(W))$ is maximized.

Remark.
There are no market frictions as the sign of the portfolio weights are unrestricted and there is no loss of return due to transaction costs or taxes.

b. Mean-Variance Formulation

Usually, the above problem is too general. Make a simplification.

Define E(W) to be the expected wealth of a portfolio, and
Var(W) to be the variance of a portfolio.

Note:

$$E(W) = \sum_{i=1}^{N} w_i Er_i$$

and

$$Var(W) = \sum_{j=1}^{N} \sum_{i=1}^{N} w_i w_j \operatorname{cov}(r_i, r_j)$$

<u>Definition: Mean-Variance Preferences</u>

E(V(W)) = function(E(W), var(W)) where the function is strictly increasing in its first argument and strictly decreasing in its second.

<u>Fact:</u> (Sufficient Conditions for Mean Variance Preferences)

1. V is quadratic
2. The returns are normally distributed.

<u>Proof:</u>

Taylor series expansion of V gives:

$$V(W) = V(E(W)) + V'(E(W))(W - E(W)) + V''(E(W))(W - E(W))^2 / 2 + \sum_{i=3}^{\infty} V^i (E(W))(W - E(W))^i / i!$$

Take expectations,

$$E(V(W)) = V(E(W)) + V''(E(W)) \operatorname{var}(W) / 2 + \sum_{i=3}^{\infty} V^i (E(W)) E((W - E(W))^i) / i!$$

See, if quadratic all terms in the sum drop out.
See, if normally distributions, the the higher order moments only depend on E(W) and Var(W).

This completes proof.

c. Mean Variance Frontier

Under mean-variance preferences, the problem is to choose a portfolio $(w_1, \cdots, w_N)$ such that

$\text{function}(E(W), \text{var}(W))$ is maximized.

It is easy to see that the optimal portfolio

$(w_1, \cdots, w_N)$ will satisfy

min var(W) subject to E(W) = r* for some r* a constant, where r* is determined by the particular utility function used.

Thus, the solution to the optimal portfolio is transformed to a QP problem.

Choose $(w_1, \cdots, w_N)$ to minimize $\sum_{j=1}^{N} \sum_{i=1}^{N} w_i w_j \text{cov}(r_i, r_j)$ subject to

$\sum_{i=1}^{N} w_i = 1$ and $\sum_{i=1}^{N} w_i Er_i = r^*$ for some r* a constant.

r* is selected later using the explicit utility function.

<u>Remarks.</u>

1. When all assets are risky, and the covariance matrix is nonsingular, much is known about this solution.

2. When all assets are risky but one, and the risky asset covariance matrix is nonsingular, much is known about this solution.

3. The Algorithmics software is a different, but related optimization problem. Different utility function. Similar constraint.