CS – 522 Computational Tools and Methods in Finance
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Lecture 5: Term Structure Models

1. Definitions

Let trading take place continuously over $[0,\tau]$.

Consider a probability space (Ω,F,Q) with filtration $(F_t : t \in [0,\tau])$ generated by n independent Brownian motions $\{W_1(t),\dots,W_n(t)\}$.

Let E(.) denote expectation with respect to Q.

All traded assets are default free. Consider zero coupon bonds with time t price P(t,T).

Forward rates are implicitly defined by

$$P(t,T) = \exp\{-\int_t^T f(t,u)du\}.$$

The forward rate f(t,T) corresponds to the riskless lending/borrowing rate one could contract at time t for period [T,T+dt].

The spot rate is defined by

$$r(t) = f(t,t).$$

A money market account is defined by

$$B(t) = \exp\{\int_0^t r(u)du\}.$$

2. Arbitrage Pricing Theorems

There are two fundamental theorems of asset pricing.

<u>Theorem 1</u>: No arbitrage

Given trading in the zeros, there is no arbitrage if and only if there exists a probability measure $\tilde{Q}$ such that:

$$\tilde{Q}(A) = 0 \Leftrightarrow Q(A) = 0 \quad \text{for all } A \in F \text{ and}$$

P(t,T)/B(t) is a $\tilde{Q}$ martingale.

<u>Theorem 2</u>: Complete markets

Given trading in the zeros and no arbitrage, then markets are complete if and only if $\tilde{Q}$ is unique.

By complete we mean that any interest rate derivative can be synthetically constructed using a dynamic trading strategy in the zero coupon bonds and the money market account.

<u>Lemma (Risk Neutral Valuation):</u>

Given no arbitrage and complete markets.
Let x(T) be a F_T measurable random variable which is squared integrable under $\tilde{Q}$.
Then, the time t cost of generating x(T) is $\tilde{E}_t(x(T)/B(T))B(t)$.

This lemma provides a technique for computing prices of interest rate derivatives.

3. Bond Dynamics

Assume that forward rates follow the stochastic process:

$$f(t,T) = f(0,T) + \int_0^T \alpha(v,T)dv + \sum_{i=1}^{n} \sigma_i(v,T)dW_i(v)$$

(technical conditions omitted).

Applying Ito's lemma one can show that (using the definition of P(t,T) in terms of f(t,u)) that

$$P(t,T) = P(0,T)\exp\{\int_0^t [r(v) + b(v,T)]dv - (1/2)\sum_{i=1}^{n}\int_0^t a_i(v,T)^2 dv$$

$$+\sum_{i=1}^{n}\int_0^t a_i(v,T)dW_i(v)$$

where

$$a_i(t,T) = -\int_t^T \sigma_i(t,v)dv$$

$$b(t,T) = -\int_t^T \alpha(t,v)dv + (1/2)\sum_{i=1}^{n} a_i(t,T)^2$$

Applying the above theorems on asset pricing, assuming the market is arbitrage free and complete implies that

There exists $f_i(t)$ for i = 1, ... , n such that

$$\alpha(t,T) = -\sum_{i=1}^{n} \sigma_i(t,T)[\phi_i(t) - \int_t^T \sigma_i(t,v)dv].$$

This is the contribution of Heath, Jarrow and Morton.

The $\phi_i(t)$ are called the market prices of risk.

Using Girsanov's theorem, one can show that under $\tilde{Q}$ the dynamics of P(t,T) are:

$$P(t,T) = P(0,T)\exp\{\int_0^t r(v)dv - (1/2)\sum_{i=1}^{n}\int_0^t a_i(v,T)^2 dv$$
$$+\sum_{i=1}^{n}\int_0^t a_i(v,T)d\tilde{W}_i(v)$$

where

$$\tilde{W}_i(t) = W_i(t) - \int_0^t \phi_i(v)dv \quad .$$

Combined with the lemma in section 2, we can now price interest rate derivatives.

4. Example (Extended Vasicek)

This section considers an important example. It is a one-factor model.

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-(\lambda/2)(T-t)}dW(t)$$

No arbitrage and market completeness implies there exists a $\phi(t)$ such that

$$\alpha(t,T) = -\sigma e^{-(\lambda/2)(T-t)}\phi(t) - 2(\sigma/\lambda)^2 e^{-(\lambda/2)(T-t)}(e^{-(\lambda/2)(T-t)} - 1)$$

Under $\tilde{Q}$ the forward rate process is:

$$df(t,T) = -2(\sigma/\lambda)^2 e^{-(\lambda/2)(T-t)}(e^{-(\lambda/2)(T-t)} - 1)dt + \sigma e^{-(\lambda/2)(T-t)}d\tilde{W}(t)$$

Consider a European call option on the bond P(t,T) with an exercise price of K and maturity date t* where 0 <= t <= t* <= T.

Let C(t) denote the call's price, then

$$C(t) = \tilde{E}_t(\max(P(t^*,T) - K,0)/B(t^*))B(t)$$
$$= P(t,T)\Phi(h) - KP(t,t^*)\Phi(h-q)$$

and

$$h = [\log(P(t,T)/KP(t,t^*)) + (1/2)q^2]/q$$
$$q^2 = (4\sigma^2/\lambda^3)(e^{-(\lambda/2)T} - e^{-(\lambda/2)t^*})^2(e^{\lambda t^*} - e^{\lambda t}).$$

Remarks:

1. Other interest rate derivatives can easily be priced.
2. The evolution in f(t,T) can be approximated via a lattice and numerical procedures akin to the binomial model used.
3. P.D.E. procedures can also be invoked.