

CS – 522 Computational Tools and Methods in Finance
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Lecture 4: Treasury Zero Curves

1. Coupon Bond Pricing

Consider a Treasury (default free) coupon bond that pays C_t dollars at times $t \in 1, 2, \dots, T$ (the life of the bond). The last payment C_T includes the principal repayment.

Let $B(t)$ represent the price of the coupon bond at time t .

Consider a Treasury zero-coupon bond that pays 1 dollar at time T .

Let $P(t,T)$ represent the time t price of the zero with payment at time T .

Theorem: Given no arbitrage and trading in the coupon bond and all the zeros,

$$B(0) = \sum_{t=1}^T C_t P(0,t).$$

Proof:

Consider the following two buy and hold portfolios, both formed at time 0.

- 1] Hold the coupon bond.
- 2] Hold C_t units of the zero with maturity t for $t = 1, 2, \dots, T$.

The payoffs of both portfolios are identical for all times from 1 to T .

Thus, the cost of both portfolios must be the same to avoid arbitrage. \checkmark

Remarks:

1. This theorem underlies the construction of Treasury Strips by the Federal Reserve Bank.
2. This theorem holds for corporate bonds, but under more stringent conditions.

2. Stripping Treasuries

The zero coupon bond prices are useful for pricing interest rate derivatives. Usually, coupon bonds trade and not the zeros. We observe the coupon bond prices, but want the

zero prices. Stripping Treasuries is the process of computing the implied zero coupon bond prices from the prices of the coupon bonds.

Let us observe N bond prices at time 0.

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ B_N \end{bmatrix} \text{ with a coupon payments matrix } C = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{1T} \\ C_{21} & & & & & \\ & & & & & \\ & & & & & \\ C_{N1} & & & & & C_{NT} \end{bmatrix}$$

In this matrix C, the bonds are ordered with the smallest maturity in row 1 up to the largest maturity in row N. The longest maturity bond has a maturity of T. This determines the number of columns in the matrix. The matrix has (almost) a lower triangular form.

In this matrix, we discard any bond such that its row in C is a linear combination of other rows. The discarded bonds can be completely priced by a no arbitrage argument. Thus, without loss of generality, all the rows in C are all linearly independent.

The purpose is to compute the zero coupon bond vector $P = [P_1 \ P_2 \ \cdot \ \cdot \ P_T]$ using the theorem, which gives: $B = C P$.

This is N equations in T unknowns.

Case 1) $N < T$ (a few bonds)

In this case there is a non-unique solution.

Example:

$$B_1 = P_1$$

$$B_2 = 2P_1 + 2P_2 + 5P_3$$

The solution is:

$$P_1 = B_1$$

$$2P_2 + 5P_3 = B_2 - 2B_1$$

Any number of P_2 and P_3 will satisfy this condition.

To get a unique solution, a number of approaches can be employed.

For example, linear interpolation can be used. This adds an equation $P_2 = (1/2)P_1 + (1/2)P_3$. Combined with the above equations, a unique solution is produced. \ddot{y}

In this situation, to get a unique solution, additional constraints need to be included. For example, linear interpolation between “missing” maturity dates is common. Polynomial approximations could also be employed.

Case 2) $N = T$.

In this case there is a unique solution: $P = C^{-1}B$.

Example:

$$B_1 = P_1$$

$$B_2 = 2P_1 + 5P_2$$

The solution is

$$P_1 = B_1$$

$$P_2 = (B_2 - 2B_1)/5$$

\ddot{y}

Case 3) $N > T$ (too many bonds)

In this case, there are no solutions or one solution. No solution means that there are arbitrage opportunities.

In the case of no solutions, to get a “best” P vector, one can solve the following optimization problem:

Find P to minimize $\sum_{i=1}^N E_i^2$ where

$$B = CP + E$$

and

$$E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_N \end{bmatrix}$$

Remarks:

1. Additional constraints could be added as desired.

2. Other objectives functions could be used.

Example:

$$B_1 = P_1$$

$$B_2 = 2P_1$$

Let $B_1 = .75$ and $B_2 = 1$. In this case there is no solution for P_1 .

There is an arbitrage since both bonds mature at time 1. The first bond pays a dollar, the second bond pays 2. Sell two of the first and buy one of the second. This portfolio brings in .5 dollars at time 0 and has no obligation at time 1.

The minimization problem is:

Choose P_1 to minimize:

$$E_1^2 + E_2^2 \quad \text{where}$$

$$.75 = P_1 + E_1$$

$$1 = 2P_1 + E_2$$

The solution is to choose P_1 to minimize $(.75 - P_1)^2 + (1 - 2P_1)^2$. Ordinary calculus gives $P_1 = (1.75/3)$.

3. Continuous Zero Curves

Given the P zero coupon bond price vector with prices for maturities 1, 2, ..., T we often desire to get prices in between the discrete dates (turn the vector into a curve as a function of maturity). Spline techniques are used in this regard.

Remark: Often desired are forward rate curves (discussed next).