CS – 522 Computational Tools and Methods in Finance
Robert Jarrow
Lecture 3: Bond Price Analytics

1. Coupon Bonds, Zero Coupon Bonds, Market Yields

Coupon Bond. A coupon bond is a financial instrument that has *promised* regular payments of C dollars (usually every 6 months) for a fixed period of time (the life of the bond) and a principal repayment of F dollars at the maturity date.

Let the coupon payments be made at times 1, 2, … , T. T is called the maturity date of the bond. The coupon payments are C dollars per period. The face value is F dollars. The coupon rate is C/F.

A zero-coupon bond is a coupon bond with $C \equiv 0$.

Corporations issue coupon bonds and governments issue zero coupon and coupon bonds.

Bond's market yield is that discrete rate y (per period) such that

$$B(0) = \sum_{t=1}^{T} \frac{C}{(1+y)^t} + \frac{C+F}{(1+y)^T}$$

where B(0) is the bond's time 0 price.

This is the internal rate of return on the bond.

<u>Remarks:</u>

1. In the bond markets, payments are usually semi-annually. The formula is adjusted as rates are stated per year. y becomes (y/2) and the intervals are of length (1/2).
2. Actual day counts are used from time 0. Time 0 corresponds to the settlement date (i.e. the date cash is exchanged). This is different from the date the order is placed (usually 1 day later).
3. The quote in the newspaper does not include accrued interest. The price paid is accrued interest plus the quoted price. The accrued interest is C[days from the last coupon payment until settlement]/[time between coupon payments].
4. The yield is a function of C, F, T.
5. The graph of y = y(T) is called the yield curve. Difficult to interpret as C, F are not the same across bonds.

2. Duration, Modified Duration, Convexity

Duration (or Macaulay's duration)

$$D(0) = \left[\sum_{t=1}^{T} \frac{tC}{(1+y)^t} + \frac{T(C+F)}{(1+y)^T} \right] \Big/ B(0) = -[\partial B(0)/\partial y](1+y)/B(0).$$

This corresponds to the average life of the bond. It is a weighted average of times the cash flows are received. The weights correspond the present value of the cash flow to the bond's price. The weights sum to one.

Modified duration.

$$D(0)_M = D(0)/(1+y) = -[\partial B(0)/\partial y]/B(0).$$

Convexity.

$$C(0) = \left[\sum_{t=1}^{T} \frac{t(t+1)C}{(1+y)^t} + \frac{T(T+1)(C+F)}{(1+y)^T} \right] \Big/ (1+y)^2 B(0) = [\partial^2 B(0)/\partial y^2]/B(0).$$

3. Risk Management

Risk management of coupon bonds follows from using a Taylor series expansion of B(0) = B(0,y) around y.

$$B(y+\Delta y) = B(y) + (\partial B(y)/\partial y)\Delta y + (\partial^2 B(y)/\partial y^2)(\Delta y)^2/2 + other.$$

$$B(y+\Delta y) = B(y) - D(0)_M B(y)\Delta y + C(0)B(y)(\Delta y)^2/2 + other.$$

Consider a portfolio of two bonds: n units of B_1 and m units of B_2 (easily generalized).

Let $V(y_1, y_2) = nB_1(y_1) + mB_2(y_2)$. Then,

$$\Delta V(y_1, y_2) = n\Delta B_1(y_1) + m\Delta B_2(y_2).$$

Substitution of duration and convexity gives:

$$\Delta V(y_1, y_2) = -nD_{M,1}B_1(y_1)\Delta y_1 - mD_{M,2}B_2(y_2)\Delta y_2 + nC_1B(y_1)(\Delta y_1)^2/2 + mC_2B(y_2)(\Delta y_2)^2/2 + other$$

Duration hedging is setting:

$-nD_{M,1}B(y_1) - mD_{M,2}B(y_2) = 0$. This gives

$-nD_{M,1}B_1(y_1)\Delta y_1 - mD_{M,2}B_2(y_2)\Delta y_2 = 0$ assuming that $\Delta y_1 = \Delta y_2$. Then, the portfolio is neutral to the first order changes in yields.

Convexity hedging is setting:

$nC_1B(y_1) + mC_2B(y_2) = 0$. This gives

$nC_1B(y_1)(\Delta y_1)^2/2 + mC_2B(y_2)(\Delta y_2)^2/2 = 0$ assuming that $\Delta y_1 = \Delta y_2$. Then, the portfolio is neutral to the second order changes in yields.

Remarks:

1. Duration and convexity hedging assumes that the yield curve shifts only in a parallel fashion. This is not empirically verified.
2. Both of these hedges ignore higher order terms.
3. This analysis ignores changes in time (t).
4. This analysis is the “traditional approach”. It is being replaced by what we study next.