

Matlab Bond Pricing Examples

(Traditional bond analytics)

1.0 Review of the definitions

The bond price B at time 0 can be priced by the following formula :

$$B = \sum_{t=1}^{T-1} \frac{C}{(1+y)^t} + \frac{C+F}{(1+y)^T} \quad (1.1)$$

where C is the coupon payment, F is the face value, and y is the yield.

Duration or Macaulay's duration (1st order) is defined by

$$D = -\left(\frac{\partial B}{\partial y}\right)(1+y) / B = \left[\sum_{t=1}^{T-1} \frac{tC}{1+y^t} + \frac{T(C+F)}{1+y^T} \right] / B \quad (1.2)$$

It is a weighted average of payment times.

Modified duration is

$$D_M = \frac{D}{(1+y)} = \left(-\frac{\partial B}{\partial y}\right) / B \quad (1.3)$$

From (1.3), we can see $\frac{\Delta B}{B} \approx -D_M \Delta y$, i.e., the relative change of bond price can be approximated by the product of the modified duration and a small shift in yield.

Convexity (2nd order) is

$$C = \frac{\partial^2 B}{\partial y^2} / B \quad (1.4)$$

2.0 Example 1 : Bond price and sensitivity (ftspex1.m)

In this example, we analyze relative importance of duration and convexity of a bond portfolio.

In the following bonds matrix, each row represents a bond and each column shows a parameter.

```
Settlement      Maturity      Face      Coupon      Number of      Basis
   date          date          value      rate      payments
Bonds = [today datenum('06/17/2010')  100      .07      2      0
         today datenum('06/09/2015')  100      .06      2      0
         today datenum('05/14/2025') 1000      .045     2      0];
yield = [0.05 0.06 0.065];
```

Coupon rate is nominal (annual) rate of coupon payments.

`prbond` is a Matlab function which returns the price p and accrued interest ai of a bond given all the parameters. The syntax is

$$[p, ai] = \text{prbond}(sd, md, rv, cpn, yld, per, basis)$$

where sd is the settlement date, md is the maturity date, rv is the par value or face value, cpn is the coupon rate, yld is the yield, per is the number of coupon periods per year, and the $basis$ is the day-count basis: 0 = actual/actual, 1 = 30/360, 2 = actual/360, 3 = actual/365.

The coupon rate and number of coupon payments are used to calculate the amount of coupon C paid during each period. Accrued interest is determined by the following formula :

$$ai = C \times \left(\frac{\text{settlement date} - \text{previous coupon date}}{\text{time between periods}} \right) \quad (2.1)$$

and the number of days in the numerator and denominator are measured differently according to the day-count basis.

$[d, m] = \text{bonddur}(sd, md, rv, cpn, yld, per, basis)$ finds the Macaulay duration d and modified duration m in years and $[pc, yc] = \text{bondconv}(sd, md, rv, cpn, yld, per, basis)$ returns the convexity for a security in periods pc and years yc .

The duration of the portfolio p_D_M and convexity of the portfolio p_c can be calculated by the following formula.

$$p_D_M = \sum_{i=1}^3 w_i D_M(i) \text{ and } p_c = \sum_{i=1}^3 w_i y_C(i) \quad (2.2)$$

where w_i is the the weight of i -th bond in the portfolio and $D_M(i)$ and $y_C(i)$ are duration and convexity of i -th bond, respectively.

Suppose there is a shift in yield curve, $dY = .002$, i.e., 20 basis points where 1 basis point is 1/100 percent or 1/10000.

Percentage change of the portfolio price (linear approximation) is

$$\text{perc}_1 = -p_D_M * dY * 100 \quad (2.3)$$

and the second order approximation is

$$\text{perc}_2 = \text{perc}_1 + \frac{p_c * dY^2 * 100}{2} \quad (2.4)$$

The approximate prices are

$$\text{price}_1 = \text{original} + \frac{\text{perc}_1 * \text{original}}{100} \quad (2.5)$$

$$\text{and price}_2 = \text{original} + \frac{\text{perc}_2 * \text{original}}{100} . \quad (2.6)$$

3.0 Example 2 : Hedging against duration and convexity (ftspex2.m)

This example constructs a bond portfolio to hedge the portfolio of example 1. We assume a long position in the portfolio in the example 1, and that three other bonds are available for hedging. We compute weights for these three other bonds in a new portfolio so that the duration and convexity of the new portfolio match those of the original portfolio.

The following are the bonds we use in hedging :

```

        Settlement      Maturity      Face      Coupon      Number of      Basis
        date           date           value     rate        payments
Bonds = [today datenum('06/15/2005')  500      .07         2           0
         today datenum('08/02/2010')  1000     .066        2           0
         today datenum('03/01/2025')   250      .08         2           0];

yield = [0.06 0.07 0.075];

```

Step 1 : Compute the prices, modified durations - dur_1 , dur_2 , and dur_3 -, and convexities - $conv_1$, $conv_2$, and $conv_3$ - of the above bonds using Matlab functions `prbond`, `bonddur`, and `bondconv`, respectively.

Step 2 : Get the weights of the bonds using the Matlab `\` operator.

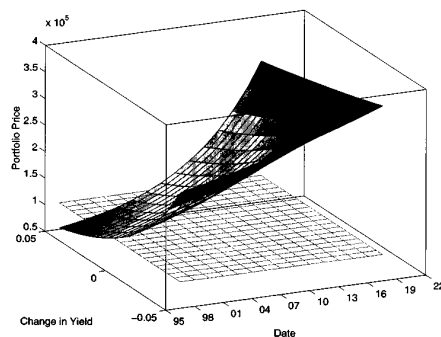
$$A = \begin{bmatrix} dur_1 & dur_2 & dur_3 \\ conv_1 & conv_2 & conv_3 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} p_{DM} \\ p_c \\ 1 \end{bmatrix} \tag{3.1}$$

and the vector w of weights is

$$w = A \backslash b \tag{3.2}$$

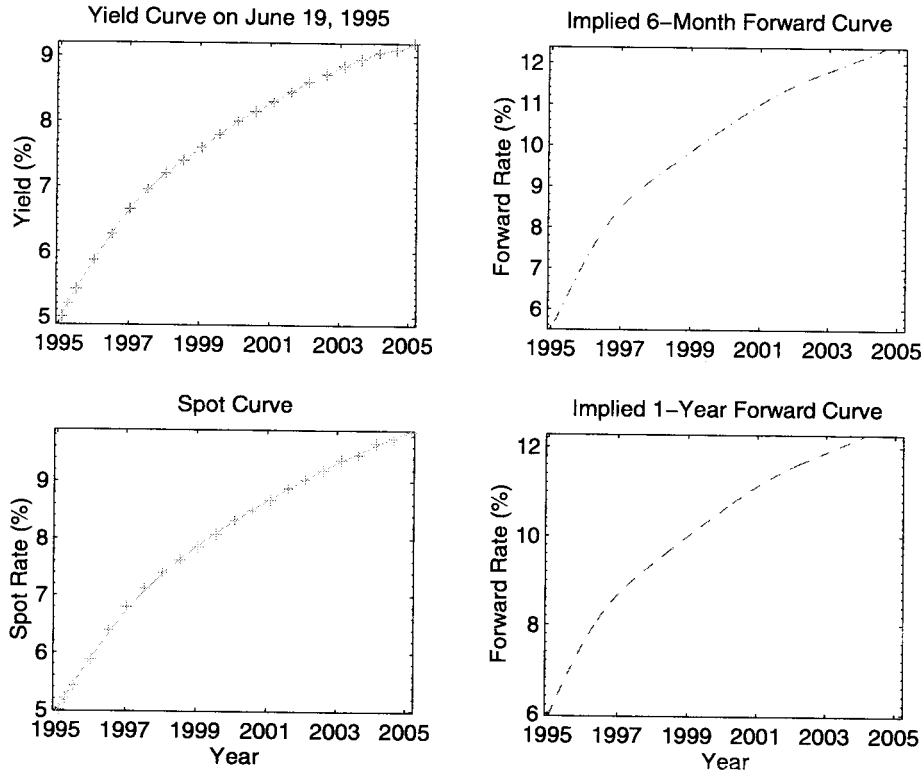
4.0 Example 3 : Visualizing the sensitivity of a bond portfolio's price to parallel shifts in the yield curve (ftspex3.m)

This example uses the Financial Toolbox bond pricing functions in a routine that takes time-to-maturity and yield as input arguments and returns the price of a portfolio of bonds. It plots the price and shows the behavior of bond prices as yield and time vary.



5.0 Yield curves and spot curves from bond data

This example takes coupon bearing bonds with prices as input and generates the following four graphs.



(1) Extracting yields

Recall

$$B = \sum_{t=1}^{T-1} \frac{C}{(1+y)^t} + \frac{C+F}{(1+y)^T} \quad (5.1)$$

The Matlab function `yldbond` returns yields when the bond prices are given. `yldbond` uses Newton's method and cubic spline interpolation.

(2) Spot rate (zero coupon yield curve)

Use bootstrapping and smoothing with splines. Find each successive spot rate by requiring that the bond of the corresponding maturity be priced correctly, given all the previous rates calculated. This process is known as the "bootstrap method" for calculating spot rates.

(3) Extracting a forward rate curve from zero coupon yield curve

Suppose y_i and f_i are the yield and the forward rate at the time T_i respectively. The relation $\exp(y_i T_i) = \exp(y_{i-1} T_{i-1}) \exp(f_i [T_i - T_{i-1}])$ gives the formula for the forward rate.

$$f_i = \frac{y_i T_i - y_{i-1} T_{i-1}}{T_i - T_{i-1}} \quad (5.2)$$

6.0 Portfolio Analysis : Heavy use of QPs and LPs

Often finding the optimal portfolio is formulated as a QP.

$$\begin{aligned} \min \quad & x^T C x \\ \text{s.t.} \quad & x^T r \geq r^* \\ & x^T e = 1 \end{aligned} \tag{6.1}$$

C is a symmetric positive definite (SPD) matrix. The first constraint is for the expected profit.

Variations

- (1) If no shorting is allowed, nonnegativity on x is required.
- (2) Sometimes there is a disjunctive constraint (much harder) :

$$x_i = 0 \text{ or } x_i \geq l_i > 0 \tag{6.2}$$

- (3) Risk aversion factor

$$\begin{aligned} \min \quad & \frac{k}{2} x^T C x - r^T x \\ & x^T e = 1 \end{aligned} \tag{6.3}$$

where k is the risk tolerance. The objective function is called a utility function.

- (4) Downside minimization over scenarios

$$\begin{aligned} \min \quad & E(\|(D^T x - \tau)_-\|) \\ & \text{expected gain over benchmark} \geq K \end{aligned} \tag{6.4}$$

where $y_- = \min(0, y)$. The constraint is linear and the optimization problem can be formulated as LP.

Solving positive definite QPs

1. The problem has a unique global solution.
2. Iterative solution
 - i) Active set method : finite algorithm
 - direct matrix factorization
 - matrix updating
 - e.g. - Matlab (to change soon)
 - CPLEX
 - ii) Asymptotic (usually interior)
 - finds a sequence of points converging to the solution
 - direct and indirect matrix solutions
 - fast convergence driven by KT conditions