CS – 522 Computational Tools and Methods in Finance
Robert Jarrow
Lecture 2: Equity Options (continued)

1. Derivation of the Black-Scholes partial differential equation

As before, let

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Suppose we consider a derivative security whose value is:

$$V(t) = V(t, S(t)).$$

Remark: This could be a call or put option (American or European).

By Ito's lemma, we can write

$$dV(t) = \partial V/\partial t \, dt + \partial V/\partial S \, dS + (1/2)\partial^2 V/\partial S^2 \, dS^2$$

Suppose we form a portfolio over [t, t+dt] consisting of the stock and derivative, with portfolio value

$P(t) = V(t) - NS(t)$ where

$$N = \partial V/\partial S.$$

Then, $dP(t) = dV(t) - NdS(t).$

Plugging in the value for dV(t), we obtain that

$$dP(t) = \partial V/\partial t \, dt + (1/2)\partial^2 V/\partial S^2 \, dS^2.$$

But $dS^2 = \sigma^2 S^2 dt$. Substitution yields

$$dP(t) = \partial V/\partial t \, dt + (1/2)\partial^2 V/\partial S^2 \, \sigma^2 S^2 dt.$$

Note that this term is deterministic. To avoid the existence of arbitrage, it must be that

$$dP(t) = rP(t)dt.$$

Making the substitutions, we get that

$$rV(t)dt - r\,\partial V/\partial S\; Sdt = \partial V/\partial t\; dt + (1/2)\,\partial^2 V/\partial S^2\; \sigma^2 S^2 dt .$$

The dt terms cancel, to yield the Black-Scholes p.d.e.

Remarks:

1. The term involving $\partial V/\partial t$ is going forward in time, if one defined (T-t) as the argument, then t is replaced by (T-t) and a minus sign appears in front of this term.

2. This p.d.e holds for any derivative. To make it well-formed, we need to specify certain boundary conditions. For example,
   for the European call, V(T,S(T)) = max[S(T)-K,0],
   for the European put, V(T,S(T)) = max[K-S(T),0].

3. This derivation is heuristic because it is done for differentials (keeping N fixed). It can be done more correctly using stochastic integrals and self-financing trading strategies.

2. Risk Management

To control the risk of an option portfolio, three considerations apply.

1. Delta Neutral Portfolios
   According to the derivation of the Black-Scholes formula, if trading took place continuously, then a delta neutral portfolio would be risk free over [t, t+dt].
   But, trading can only be done discretely.

2. Gamma Neutral Portfolios
   If trading takes place continuously, then dS and dS^2 are of the same magnitude, hence need to hedge movements in dS^2. This is a gamma neutral position.
   Sometimes called jump or gap risk.

3. Vega Neutral Portfolios
   The volatility is assumed constant and known. If this is not true, then the model is misspecified. Hence, a vega neutral portfolio is trying to eliminate model error.