

**CS – 522 Computational Tools and Methods in Finance**  
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**Lecture 1: Equity Options**

**1. Definitions**

Equity. The common stock of a corporation. Traded on organized exchanges (NYSE, AMEX, NASDAQ). A common stock issues dividends, has voting rights and a resale value.

Options. A financial security issued against the common stock of a corporation. Two types: calls and puts. Two types of these: European and American. Options are traded on organized exchanges (CBOE).

European Call option gives its owner the right to **buy** a common stock at a fixed price (strike price, exercise price) on a fixed future date (expiration date, maturity date). The cost of this option is called the option's premium. Can buy (go long) or sell (go short or write) call options.

European Put option gives its owner the right to **sell** a common stock at a fixed price (strike price, exercise price) on a fixed future date.

Options are not dividend protected, i.e. owner of the option does not receive the dividends on the stock.

Discuss American type options later.

**2. Notation**

Let  $S(t)$  be the time  $t$  price of a stock.

Let  $d(t)$  be the time  $t$  dividend.

Let  $c(t) = c(t : T, K)$  be the time  $t$  price of a European call option with strike  $K$  and expiration date  $T$  on the stock  $S(t)$ .

Let  $p(t) = p(t : T, K)$  be the time  $t$  price of a European put option with strike  $K$  and expiration date  $T$  on the stock  $S(t)$ .

Let  $r$  be the spot rate of interest (per unit time) for riskless investment over  $t, t + \Delta$  for an arbitrary  $t$  (constant across time). Continuous compounding.

Let  $B(t, T)$  be the time  $t$  value of a sure dollar paid at time  $T$ .

$B(t, T) = \exp(-r(T-t))$ . Term Structure of Interest Rates (return to later on in the course).

**3. Payoff Diagrams**

European Call Option

Boundary condition  $c(T) = \max ( S(T) - K, 0 )$ .

European Put Option

Boundary condition  $p(T) = \max ( K - S(T), 0 )$ .

#### 4. Market Structure

Assumptions: Frictionless markets ( no transaction costs, no taxes, no bid/ask spreads, no restrictions on short sales, etc.), competitive markets (traders act as price takers), no arbitrage opportunities (to define later ).

#### 5. Put – Call Parity

No dividends on the stock.

$$c(t : T, K ) = p(t: T, K ) + S(t) - KB(t, T)$$

Proof:

At time 0 consider the following portfolios.

Portfolio 1: buy call option. Payoff at time T.

$$c(T) = \max ( S(T) - K, 0 )$$

Portfolio 2: buy put option, buy stock, sell K bonds with maturity T. Payoff at time T.

$$p(T) + S(T) - K = \max ( K - S(T), 0 ) + S(T) - K = \max ( S(T) - K, 0 )$$

Same time T values. Both portfolio 1 and 2 have no intermediate cash flows. To avoid **arbitrage**, both portfolios must have the same time t cost. This is the result. ■

With dividends this result changes (must include dividends to the position that holds the stock).

#### 6. American options

American call option. Like the European, except it can be exercised at any time from purchase until the maturity date.

American put option. Like the European, except it can be exercised at any time from purchase until the maturity date.

This added privilege gives the American option more value than the European.

Let  $C(t) = C(t : T, K)$  be the American call's time  $t$  value.

Let  $P(t) = P(t : T, K)$  be the American put's time  $t$  value.

Fact 1:  $c(t) \leq C(t)$  . Same strike and maturity.

Fact 2:  $p(t) \leq P(t)$  . Same strike and maturity.

Fact 3:  $\max ( S(t) - K, 0 ) \leq C(t)$

Fact 4:  $\max ( K - S(t), 0 ) \leq P(t)$

Dividends can influence the value of an American call and put option. May want to exercise the option prior to maturity.

Example 1: (call)

Let  $t \leq T - 1 \leq T$ .

Let there be no dividends except at time  $T - 1$ .

Let  $d(T-1)$  be known and equal to  $S(T-1-\epsilon) - \epsilon$ , where  $\epsilon$  is a small amount. Then,  $S(T-1) = \epsilon$ .

Let  $\text{Prob}(S(T) \geq K) = 0$  after dividend.

Let  $K \leq S(t)$  (said to be in the money). If exercise at  $t$ , call worth at least  $S(t) - K$ . If wait until  $T$ , call will be worthless.

Remark: Will want to exercise (or not) at time  $T - 1$ .

European call on this stock would have zero value.      End example.

Example 2: (put)

Let  $t \leq t^* \leq T$ .

No dividends on the stock at all.

Suppose that at time  $t^*$  there is a 50 percent chance that the stock will be worth  $S(t^*) = \epsilon$  where  $\epsilon$  is small.

At time  $t^*$  will want to exercise the put if  $S(t^*) = \epsilon$ . Additional gain to waiting is  $\epsilon$  but could lose profits of  $K - \epsilon$  if wait to exercise. The reason is that the stock price could go up again.

European put will have a different value, because must wait until T to exercise (or not). End example.

Remarks:

1. European and American options have different values (in general).
2. Theorem: A European call and an American call have identical values if there are no dividends paid on the stock over the life of the option.
3. This theorem is not true for puts (see example 2).
4. Remarks 1 – 3 imply a different valuation technique is needed for American and European options.

## 7. The Binomial Model

This is a model for pricing options. It works by synthetically constructing an option from the stock and riskless investment. The cost of the synthetic option is called the arbitrage free price of the traded option.

Stock Price Evolution (Binomial Lattice)

Discrete times  $0, 1, 2, \dots, T$

No dividends.

$S(0) = S > 0$ .

$$S(t+1) = \begin{cases} S(t)U & \text{with probability } q \\ S(t)D & \text{with probability } 1 - q \end{cases}$$

Assume that  $U > R \equiv \exp(r) > D$ . This is a no arbitrage condition.

A random walk. Easily generalized.

Theory (via an example).

Consider a European call on the stock with strike price K and maturity time 1.

Price by backward induction. Create the option synthetically using the stock and riskless investment.

Suppose  $SU > K > SD$ .

Time 1: The option's value is

$$c(1) = \max(S(1) - K, 0) = \begin{cases} SU - K & \text{if U occurs} \\ 0 & \text{if D occurs.} \end{cases}$$

Time 0: Form a portfolio with  $m$  shares of the stock and investing  $B$  dollars risklessly such that

Initial cost is  $mS + B$ .

Time 1 value state U :  $mSU + BR$

Time 1 value state D :  $mSD + BR$

Choose  $m$  and  $B$  such that

$$mSU + BR = SU - K$$

$$mSD + BR = 0.$$

Two equations in two unknowns. Solution is:

$$m = \frac{((SU - K) - 0)}{S(U - D)} \quad \text{and} \quad B = \frac{(0 - mSD)}{R}$$

This portfolio duplicates the option's time 1 payoff. The no **arbitrage** value of the option is the cost of the time 0 portfolio.

Hence,

$c(0) = mS + B$  with these solutions. Algebra gives

$$c(0) = [\pi(SU - K) + (1-\pi)0]/R \quad \text{where} \quad \pi = [R - D]/[U - D].$$

This price does not depend on  $q$ . Note that the  $\pi$  can be interpreted as a probability, strictly between zero and one. It is called a pseudo (risk neutral, martingale) probability. End of example.

### Remarks.

1. Recursive formulation. Replace  $SU - K$  with  $c(1:U)$  and  $0$  with  $c(1:D)$ . Replace  $S$  with  $S(0)$ .

$$c(0) = [\pi c(1:U) + (1-\pi)c(1:D)]/R \quad \text{where} \quad \pi = [R - D]/[U - D].$$

2. Extends to multiple time periods. Extends to dividends.

3. Binomial model (strike  $K$ , maturity  $T$ ). Multiple time periods.

$$c(0) = \sum_{j=0}^T \binom{T}{j} \pi^j (1-\pi)^{T-j} \max(SU^j D^{T-j} - K, 0) / R^T.$$

4. European put identical except replace the boundary condition with  $\max(K - S(T), 0)$ .

The American call valuation problem is different. At each node in the lattice, must decide on early exercise or not.

In the recursive formulation, replace  $c(1:U)$  with  $\max(c(1:U), S(0)U - K)$  and  $c(1:D)$  with  $\max(c(1:D), S(0)D - K)$ . This is the maximum of alive (value as reflected in remaining lattice) or dead (value if exercised). Same  $\pi$  applies.

## 8. Black Scholes Formula

There are many ways to derive the Black Scholes formula. I will take the simplest. Same structure as before, except for a continuous time model. No dividends.

The evolution of the stock price is

$dS(t) = S(t)\{\mu dt + \sigma dW(t)\}$  where  $W(t)$  is a Brownian motion initialized at zero.  $\mu$  and  $\sigma$  are constants.

This is called a geometric Brownian motion. This is a stochastic differential equation.

Take  $\log(S(t))$ . Using Ito's lemma, we can write this as:

$$d \log S(t) = \mu dt - (1/2)\sigma^2 dt + \sigma dW(t)$$

Integration yields

$$S(t) = S(0) \exp\{\mu t - (1/2)\sigma^2 t + \sigma W(t)\}.$$

Thus,  $S(t)$  is lognormally distributed with

$$E(\log S(t) / S(0)) = \mu t - (1/2)\sigma^2 t$$

and

$$Var(\log S(t) / S(0)) = \sigma^2 t.$$

$\sigma$  is called the stock's volatility.

To obtain the Black-Scholes formula, the idea is to approximate this geometric Brownian motion with a binomial distribution.

Let the time periods 0, 1, 2, ..., T correspond to a partition of the continuous time interval [0,T] in units of size  $\Delta$ .

Let

$$U = \exp\{\mu\Delta - (1/2)\sigma^2\Delta + \sigma\sqrt{\Delta}\}$$

and

$$D = \exp\{\mu\Delta - (1/2)\sigma^2\Delta - \sigma\sqrt{\Delta}\}$$

Then  $\log S(t)/S(0)$  will converge to the geometric Brownian motion process given above as  $\Delta \rightarrow 0$  (a simple proof by simulation).

The binomial call option model

$$c(0) \rightarrow S(0)N(d_1) - KB(0,T)N(d_2)$$

where  $N(\cdot)$  is cumulative normal

$$d_1 = \{\log(S(0)/KB(0,T)) + \sigma^2 T / 2\} / \sigma\sqrt{T}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

A simple proof by showing the approximation is valid.

Remarks.

1. This formula is the Black-Scholes model.
2. This formula does not depend on  $\mu$ .
3. By analogy with the binomial model, the synthetic call is:

$$c(0) = mS(0) + B.$$

Here, the number of shares in the stock and riskless investing are:

$$m = N(d_1) = \partial c(0) / \partial S(0) \leq 1$$

and

$$B = -KB(0,T)N(d_2) \leq 0.$$

Note that B is negative, which implies borrowing.

4. m is called the option's delta. The synthetic call is obtained from holding the portfolio in remark 3 for an instant, and then it needs to be rebalanced.
5. Other partial derivatives are used:

Gamma is the partial derivative of delta with respect to  $S(0)$   
Vega is the partial derivative of  $c(0)$  with respect to  $\sigma$ .

### **Reference**

Jarrow, R. and S. Turnbull, 1996, *Derivative Securities*, Southwestern Publishers:  
Cincinnati, Ohio.