COM S 783 / OR&IE 634 Approximation Algorithms	Fall Semester, 2003
Lecture 9: Sept 29, 2003	
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1 LP for Multiway cut [1]

Last time we set up the following LP for the Multiway cut problem:

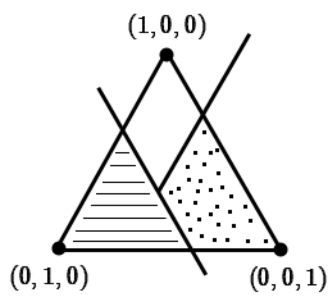
$$
\begin{array}{lll}
min & \frac{1}{2}\sum_{uv} ||x^u - x^v||_1 \cdot c_{uv} & \\
 & & \\
s.t. & x^i = e^i & \forall terminal\ i \\
 & \sum_j x^u_j = 1 & \forall u \\
 & x^u \geq 0 & \forall u
\end{array}
$$

This LP maps points of the graph into the simplex $\Delta_k = \{x \in \mathbb{R}^k | x \geq 0, ||x||_1 = 1\}$. For example, for $k = 3$, the simplex is an equilateral triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. It can be checked that every feasible solution to this LP is a feasible solution to the previous LP (the one with arbitrary metrics), but not the other way around.

2 A rounding procedure

The rounding procedure presented here is not the best rounding procedure that is known, but it's simpler. For $k = 3$, a procedure is known that matches the integrality gap (it will be presented later in this lecture), but not for other values of k.

The idea of the rounding method is to subdivide the simplex into regions, assigning each region to a vertex. Then the edges going between regions are included in the cut. We will make cuts using $(k-2)$-dimensional hyperplanes parallel to the simplex facets, which, in the case of $k = 3$, are just lines parallel to the triangle's sides.



Such a cut is defined by a parameter $0 \leq \rho \leq 1$, such that all points on this hyperplane have the same coordinate corresponding to that vertex: $x_j = \rho$. Then the set $\{u | x_j^u \geq \rho\}$ is assigned to terminal j. At the subsequent steps, only the nodes remaining in the graph are cut.

The rounding algorithm is:

- Pick a random permutation π of the terminals
- Pick uniformly at random $0 \leq \rho \leq 1$
- For $i = 1...(k-1)$
    - assign to terminal π_i all the unassigned nodes in $\{u | x_{\pi_i}^u \geq \rho\}$

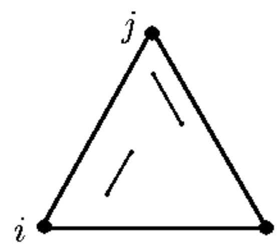
As can be seen from the analysis that follows, the only two requirements that we have for the random permutation are that (1) every pair of terminals appears in the two orders with equal probability, and (2) each terminal is equally likely to be the last one in the permutation. So it is sufficient to choose randomly from a small number of permutations, instead of all possible ones. Also, there is a polynomial number of values for ρ that produce distinct cuts. As a result of these two facts, the algorithm can be derandomized by trying all possibilities for the relevant π and ρ.

Lemma 1. Without loss of generality, we may assume that for all $uv \in E$, x^u and x^v differ in at most two coordinates.

The details of the proof are omitted here. The general idea is that because L_1 norm is additive, by inserting dummy vertices, we can convert an edge whose endpoints differ in more than two coordinates into a path consisting of edges whose endpoints differ by exactly two coordinates. The objective function value is maintained by this process. The sum on probabilities that the new smaller edges are cut is an upper bound on the probability that the original edge is cut. It is not necessarily equal because in the event that two shorter edges are cut, the longer edge is cut only once.

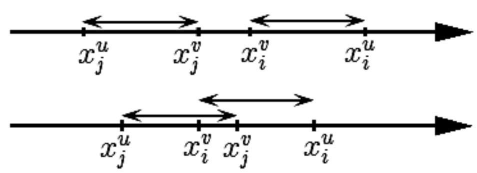
Notice that, because the sum of the coordinates is equal for any two points, if u and v differ in one coordinate, then they must differ in two. So if $x_i^u > x_i^v$, then $x_j^u < x_j^v$ and $x_i^u - x_i^v = x_j^v - x_j^u = \frac{1}{2}||x^u - x^v||_1$.

Now all edges of the graph are parallel to facets of the simplex.



As a result, there are only two possible cuts that would cut a given edge. Consider the following algorithm: Pick a random permutation of the terminals and a random ρ for each terminal, then make the cuts in the order of the permutation. In this algorithm, we have two chances to cut an edge, each of which is proportional to its length. This is a $2 - \frac{1}{k}$ approximation, as before. So using the same ρ for each terminal is important for improving the approximation factor.

Consider an edge uv and two coordinates (i, j) in which u and v differ. Suppose wlog that $x_i^u > x_i^v$ and $x_j^v > x_j^u$. Then there are two interesting possibilities for the relative location of the four coordinates on a number line (the 4 marked regions are all of the same length):



The other two possibilities are isomorphic to these up to renaming $i \leftrightarrow j$ or $u \leftrightarrow v$.

When can ρ cut uv? We will ignore the probability that both u and v are assigned to a terminal that occurs before i and j in the permutation, as that only reduces the probability that uv is cut. There are two cases, each of which occurs with probability $\frac{1}{2}$: $\pi(j) < \pi(i)$ and $\pi(i) < \pi(j)$. Suppose j occurs before i in the permutation. Then

$$\begin{aligned} Pr[j \ cuts \ uv|\pi(j) < \pi(i)] &\leq \frac{1}{2}||x^u - x^v||_1 \\ Pr[i \ cuts \ uv|\pi(j) < \pi(i)] &\leq \frac{1}{2}||x^u - x^v||_1 \end{aligned}$$

and

$$Pr[uv \ is \ cut|\pi(j) < \pi(i)] \leq ||x^u - x^v||_1$$

However, if i is before j, there is only one chance of cutting uv. This is because if ρ is in the correct range for j to cut uv, then both u and v are assigned to i before j has a chance to cut the edge. So in this case

$$Pr[uv\ is\ cut|\pi(i) < \pi(j)] \ \leq \ \frac{1}{2}||x^u - x^v||_1$$

And the overall probability

$$Pr[uv\ is\ cut] \ \leq \ \frac{3}{4}||x^u - x^v||_1 \ = \ \frac{3}{2}dist(u, v)$$

In fact, since there is no cut for the last terminal,

$$Pr[uv\ is\ cut] \ \leq \ (\frac{3}{2} - \frac{1}{k})dist(u, v)$$

This bound is not tight for any k. E.g, for $k = 3$, this gives a factor of $\frac{7}{6}$. The integrality gap for this relaxation is $\frac{12}{11}$ for the case $k = 3$, and the asymptotic integrality gap is known to be in $[\frac{8}{7}, 1.349)$ [4, 3].

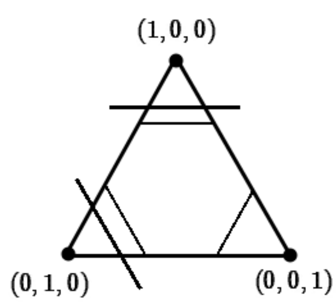
3 Case $k = 3$ [2, 4]

We will now present a rounding algorithm that achieves the ratio equal to the LP integrality gap. It uses a similar technique of cutting parallel to the triangle sides, but a different probability distribution. A cut is defined by a random permutation π (chosen uniformly) and two values $0 \leq \rho_1, \rho_2 \leq 1$. With probability p, the ρ's are chosen from the "ball cut" distribution, and with probability $(1-p)$, they are chosen from the "corner cut" distribution. p is optimized to get the best expected solution cost.

Conceptually, for this problem it is useful to divide the simplex into four regions, three of which are corners and the middle one is a hexagon. Points in a "corner" corresponding to terminal i are the ones with $x_i \geq \frac{2}{3}$.

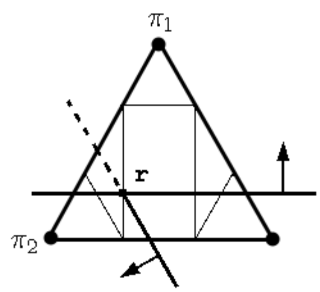
3.1 Corner cuts

Here ρ_1 and ρ_2 are chosen uniformly in $[\frac{2}{3}, 1]$. The third corner (π_3) is not cut.



3.2 Ball cuts

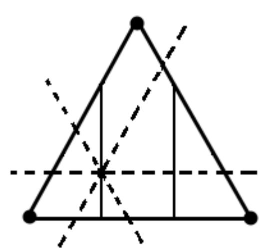
For a ball cut, one of two vertical lines shown in the figure is chosen at random, and then a point r on that line is also chosen uniformly at random. The two cuts are then made through that point: one separating a region for the first vertex in the permutation, and another separating a region for the second vertex. Then $\rho_1 = r_{\pi_1}$, and $\rho_2 = r_{\pi_2}$.



3.3 Analysis

We want to talk about edges that lie either entirely in a corner or entirely in the hexagon. So if some edge crosses the boundary, we insert a new vertex on that edge at the boundary. We also assume that edges don't cross the vertical lines that we use for ball cuts. Notice that an edge in the hexagon can only be cut by a ball cut, whereas an edge in a corner can be cut by either.

Given the point r, consider 6 "rays" eminating from r that can potentially be used in the cut. Only 3 of them will actually be used, so by symmetry each one has probability $\frac{1}{2}$ of being chosen.



Given an edge uv in the hexagon and the vertical line l on which r will lie, uv has the possibility of being cut by only two of the rays. This is because of the three rays that lie on the same side of l as uv, one is parallel to uv, and therefore cannot cut it.

Notice that r_i is distributed uniformly on $[0, \frac{2}{3}]$ for all i. This is easiest to see for the coordinate that corresponds to the top vertex on our figure, but is also true for the other coordinates. Because everything is symmetric, we will analyze only one direction.

So, for uv in the hexagon,

$$Pr[uv\ is\ cut|we\ do\ a\ ball\ cut] \ \leq \ \frac{d(u,v)}{2/3} \cdot 2 \cdot \frac{1}{2} \ = \ \frac{3}{2}d(u,v)$$

Division by 2/3 comes from the distribution of r_i, 2 represents the two possible rays that can cut uv, and $\frac{1}{2}$ is the probability of a ray to be chosen.

If uv is a corner edge and we do a corner cut, uv is cut with probability $3 \cdot \frac{2}{3} \cdot d(u,v)$, where 3 comes from the range $[\frac{2}{3}, 1]$ of ρ, and $\frac{2}{3}$ is the probability that we cut at that corner (remember that only two of three corners are cut). If we do a ball cut, only one ray from a given line l can cut uv: of the remaining two rays, one is parallel to uv, and the other does not go into uv's corner. So in that case the probability of cutting uv is $\frac{1}{2} \cdot \frac{3}{2} \cdot d(u,v)$, where $\frac{1}{2}$ is the probability of the ray being chosen, and $\frac{3}{2}d(u,v)$ is the probability that r is in the right place to cut uv.

To summarize, if p is the probability of choosing ball cuts over corner cuts, then

$$\begin{aligned} Pr[hexagon\ edge\ uv\ is\ cut] \ &\leq \ p \cdot \frac{3}{2} \cdot d(u,v) \\ Pr[corner\ edge\ uv\ is\ cut] \ &\leq \ (1-p) \cdot 2 \cdot d(u,v) + p \cdot \frac{3}{4} \cdot d(u,v) \end{aligned}$$

Choosing $p \in [0,1]$ to minimize the maximum of these two expressions yields $p = \frac{8}{11}$ and approximation ratio of $\frac{12}{11}$.

References

[1] G. Calinescu, H. Karloff, and Y. Rabani. An improved approximation algorithm for multiway cut. STOC 1998.

[2] W. H. Cunningham and L. Tang. Optimal 3-terminal cuts and linear programming. IPCO '99.

[3] A. Freund and H. Karloff. A lower bound of 8/(7+1/(k-1)) on the integrality ratio of the Calinescu-Karloff-Rabani relaxation for multiway cut. Information Processing Letters 75(1-2):43-50, 2000

[4] D. R. Karger, P. Klein, C. Stein, M. Thorup, and N. E. Young. Rounding algorithms for a geometric embedding of minimum multiway cut. STOC 1999