Lecture 18: Nov 10, 2003

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1 Facility Location - Primal-Dual Schema [1]

1.1 LP and Dual Formulation

The relaxed LP of Facility Location is

$$\begin{array}{lll} \min & \sum_{i \in F} f(i) \cdot y_i + \sum_{i \in F} \sum_{j \in C} c(i,j) \cdot x_{ij} & \\ s.t. & \sum_{i \in F} x_{ij} \geq 1 & \forall j \in C \\ & x_{ij} \leq y_i & \forall i \in F, \forall j \in C \\ & x, y \geq 0 & \end{array}$$

The Dual of the above LP is

$$\begin{array}{lll} \max & \sum_{j \in C} \alpha_j & \\ s.t. & \alpha_j - \beta_{ij} \leq c(i,j) & \forall i \in F, \forall j \in C \\ & \sum_{j \in C} \beta_{ij} \leq f(i) & \forall i \in F \\ & \alpha, \beta \geq 0 & \end{array}$$

The complementary slackness conditions are:

- Primal CSC
  - $x_{ij} > 0 \Rightarrow \alpha_j - \beta_{ij} = c(i,j)$
  - $y_i > 0 \Rightarrow \sum_{j \in C} \beta_{ij} = f(i)$
- Dual CSC
  - $\beta_{ij} > 0 \Rightarrow y_i = x_{ij}$
  - $\alpha_j > 0 \Rightarrow \sum_{i \in F} x_{ij} = 1$

The algorithm will maintain the primal CSC, and relax the dual CSC.

1.2 Interpretation of the Dual

The dual variable α_j is the price paid by client j, and out of α_j, β_{ij} is the part of the payment given to open facility i. Whenever $\sum_{j \in C} \beta_{ij} = f(i)$, we can open facility i.

The difference $\alpha_j - \beta_{ij}$ is the payment left from j after paying to facility i, and it can be used to pay for connection. It is bounded by $c(i,j)$, and $\alpha_j - \beta_{ij} = c(i,j)$ implies connecting j to i.

1.3 The Algorithm

- Initialize $\alpha = 0$, $\beta = 0$.
- Raise α_j at a uniform rate.
  - When for some i, j, $\alpha_j = c(i,j)$, start raising β_{ij} along with α_i. If i is already open, connect j to i, and "freeze" α_j.

    We say that ij is tight at this time, and from now on, $\alpha_j - \beta_{ij}$ will keep being $c(i,j)$.
  - When for some i, $\sum_{j \in C} \beta_{ij} = f(i)$,
    * if i is not open, then open i;
    * for all the unconnected j, s.t. ij is tight, connect j to i.

    After that, we cannot raise these β_{ij} any more, and therefore α_i and β_{ij} (for all tight ij) are "frozen".

Remark 1 Eventually, all clients will get connected.

If j is not connected, there are only two cases:

- α_j keeps increasing, until some ij becomes tight.
- α_j, β_{ij} increase together, and finally some facility is open.

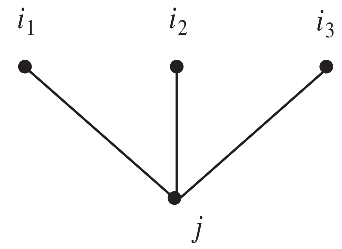
Neither of these two cases is going to last for ever.

Remark 2 If some events happen at the same time, deal with them in arbitrary order. An important issue is that when a client j gets tight with two open facilities i and i' simultaneously, we will simply connect j to one of them, and pretend that the other is not tight.

Remark 3 The algorithm generates dual feasible solution, and primal feasible solution. During the execution of the algorithm, primal CSC hold, since once a dual constraint becomes tight, it keeps to be tight.

1.4 Discussion and Analysis

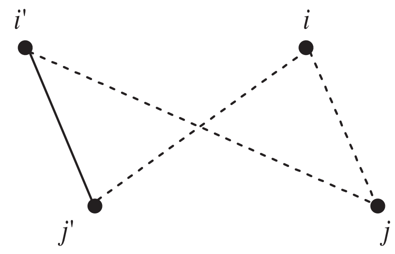
Sometime, a client j may have more than one facilities get tight simultaneously, then j pays for the opening cost of several facilities. However, in Dual, we subtract β_{ij} separately from α_j for different i. So dual CSC do not hold.



Consider a graph H on F, there is an edge ii' in H, iff there is client j that paid for both i and i', that is, ij & $i'j$ are both tight. We take in H a lexicographically (in the order of opening) first maximal independent set. It works as a clean-up procedure, that is, every time we take a facility i in the order of opening, we remove all the i' that has edge ii' in H. Since we do the clean-up in the same order as the algorithm, we can do it on the fly.

Now let us analyze how the clean-up affects connection cost. With the clean-up in effect, any client j has at most one edge ij, s.t. ij is tight, and i is open, since any other facility i' with tight $i'j$, is not picked.

On the other hand, since we force some facilities not to open, it is possible that some client j is unable to connect to any facility i, s.t. ij is tight. We will consider this situation now. Suppose we have facilities i, i', clients j, j', s.t. $ij, ij', i'j'$ are all tight, but $i'j$ is not tight, and i' is before i. We will choose to open i', not i, and have to connect j to i'.



Let α_j^f denote the contribution of α_j to opening facility, and α_j^c denote the contribution of α_j to connecting. Note that there is at most one facility to open for any client, according to the clean-up procedure.

Now we will analyze the two kinds of clients.

- For client j', it connects to i', where $i'j'$ is tight.

$$\begin{array}{rcl} \alpha_{j'}^f & = & \beta_{i'j'} \\ \alpha_{j'}^c & = & c(i', j') \\ \alpha_{j'} & = & \alpha_{j'}^f + \alpha_{j'}^c \end{array}$$

- For client j, it has to connect to i', where $i'j$ is not tight.

$$\begin{array}{rcl} \alpha_{j'}^f & = & 0 \\ \alpha_{j'}^c & = & \alpha_j \end{array}$$

We will show that the facility cost of our solution is in fact equal to the contribution of the dual solution to opening facilities (there is no approximation factor), and connection cost will be bounded by a factor times the connection contribution of the dual solution (here comes the approximation factor).

Consider the opened facility i', we have

$$f(i') = \sum_{j'' \text{ assigned to } i'} \alpha_{j''}^f$$

since when we open i', we have

$$\sum_{j'' \in C} \beta_{i'j''} = f(i')$$

So there is no approximation factor for opening cost.

Now let us take a look at connection cost. Clients like j' are able to pay for there connection. However, $i'j$ is not tight, so $\alpha_j^c = \alpha_j$ is not able to pay for connecting j to i'.

According to triangular inequality,

$$c(i', j) \leq c(i, j) + c(i, j') + c(i', j')$$

If we can show that $c(i, j), c(i, j'), c(i', j') \leq \alpha_j$, then we will have $c(i', j) \leq 3\alpha_j$, and hence have an approximation factor of 3. It is easy to see $c(i, j) \leq \alpha_j$, since ij is tight. So we only need to consider $c(i, j')$ and $c(i', j')$:

Denote the time of opening i' by t_1, and that of (temporarily) opening i by t_2. t_1 and t_2 are both measured by the amount of raise of α, since we suppose they are increased in a uniform rate. i' is opened before i, so $t_1 \leq t_2$.

We also have $\alpha_j \geq t_2$, since it is impossible for α_j to stop raising before i is temporarily open.

Claim 1. $c(i, j'), c(i', j') \leq t_2$

Proof: Suppose not. At the time when ij' or $i'j'$ becomes tight, both i and i' should have already been open. So according to **Remark 2** of the algorithm, we will only view one of them is tight, not both. It leads to a contradiction with the assumption that both ij' and $i'j'$ are tight. $\square$

Finally, we will give the approximation ratio of the algorithm. Let F' be the set of open facilities, and $g(j)$ be the facility that j gets assigned to in the end.

$$\begin{aligned} 3 \cdot OPT &\geq 3\sum_j \alpha_j \\ &= 3\sum_j (\alpha_j^f + \alpha_j^c) \\ &\geq 3\sum_{i\in F'} f(i) + \sum_{j\in C} c(g(j), j) \\ &\geq \sum_{i\in F'} f(i) + \sum_{j\in C} c(g(j), j) \end{aligned}$$

So it is a 3-approximation algorithm.

Remark We can apply this algorithm to k-Median problem, just as what we have done for Price Collecting Steiner Tree problem and k-MST problem. We will first compute two solutions, one of which is smaller than k, and the other is bigger than k, then combine them with factor of 2. So we will have a 6-approximation algorithm for k-Median. A better analysis[2] gives an approximation guarantee of 4.

References

[1] K. Jain and V. Vazirani. Primal-dual approximation algorithm for metric facility location and k-median problems. FOCS '99.

[2] M. Charikar and S. Guha. Improved combinatorial algorithms for the facility location and k-median problems. FOCS '99.