Equational theories of tropical semirings

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Abstract

This paper studies the equational theories of various exotic semirings presented in the literature. Exotic semirings are semirings whose underlying carrier set is some subset of the set of real numbers equipped with binary operations of minimum or maximum as sum, and addition as product. Two prime examples of such structures are the $(\max, +)$ *semiring* and the *tropical semiring*. It is shown that none of the exotic semirings commonly considered in the literature has a finite basis for its equations, and that similar results hold for the commutative idempotent weak semirings that underlie them. For each of these commutative idempotent weak semirings, the paper offers characterizations of the equations that hold in them, decidability results for their equational theories, explicit descriptions of the free algebras in the varieties they generate, and relative axiomatization results.
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MSC: 08A70; 08B05; 03C05; 68Q15; 68Q70

Keywords: Equational logic; Varieties; Complete axiomatizations; Relative axiomatizations; Tropical semirings; Commutative idempotent weak semirings; Convexity; Exponential time complexity

1. Introduction

Exotic semirings, i.e., semirings whose underlying carrier set is some subset of the set of real numbers $\mathbb{R}$ equipped with binary operations of minimum or maximum

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as sum, and addition as product, have been invented and reinvented many times since
the late 1950s in various fields of research. This family of structures consists of
semirings whose sum operation is idempotent—two prime examples are the $(\max, +)$
semiring

$$(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$$

(see [5, Chap. 3] for a general reference), and the *tropical semiring*

$$(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$$

introduced in [35]. (Henceforth, we shall write $\vee$ and $\wedge$ for the binary maximum
and minimum operations, respectively.) Interest in idempotent semirings arose in the
1950s through the observation that some problems in discrete optimization could be lin-
earized over such structures (see, e.g., [10] for some of the early references and [40]
for a survey). Since then, the study of idempotent semirings has forged productive
connections with such diverse fields as, e.g., performance evaluation of manufactur-
ing systems, discrete event system theory, graph theory (path algebra), Markov deci-
sion processes, Hamilton-Jacobi theory, asymptotic analysis (low temperature asymp-
totics in statistical physics, large deviations), and automata and language theory (au-
tomata with multiplicities). The interested reader is referred to [14] for a survey of
these more recent developments, and to [12] for further applications of the $(\max, +)$
semiring. Here we limit ourselves to mentioning some of the applications of vari-
ations on the tropical semiring in automata theory and the study of formal power
series.

The tropical semiring $(\mathbb{N} \cup \{\infty\}, \wedge, +, \infty, 0)$ was originally introduced by Simon
in his positive solution (see [35]) to Brzozowski's celebrated finite power property
problem—i.e., whether it is decidable if a regular language L has the property that, for
some $m \geqslant 0$,

$$L^* = 1 + L + \cdots + L^m.$$

The basic idea in Simon's argument was to use automata with multiplicities in the
tropical semiring to reformulate the finite power property as a Burnside problem. (The
original Burnside problem asks if a finitely generated group must necessarily be finite if
each element has finite order [7].) The tropical semiring was also used by Hashiguchi
in his independent solution to the aforementioned problem [16], and in his study of
the star height of regular languages (see, e.g., [17–19]). (For a tutorial introduction
on how the tropical semiring is used to solve the finite power property problem, we
refer the reader to [32].) The tropical semiring also plays a key role in Simon's study
of the nondeterministic complexity of a standard finite automaton [37]. In his thesis
[26], Leung introduced topological ideas into the study of the limitedness problem for
distance automata (see also [27]). For an improved treatment of his solutions, and
further references, we refer the reader to [28]. Further examples of applications of the
tropical semiring may be found in, e.g., [24,25,36].

The study of automata and regular expressions with multiplicities in the tropical
semiring is by now classic, and has yielded many beautiful and deep

results—whose proofs have relied on the study of further exotic semirings. For example, Krob has shown that the equality problem for regular expressions with multiplicities in the tropical semiring is undecidable [22] by introducing the *equatorial semiring* $(\mathbb{Z} \cup \{\infty\}, \wedge, +, \infty, 0)$, showing that the equality problem for it is undecidable, and finally proving that the two decidability problems are equivalent. Partial decidability results for certain kinds of equality problems over the tropical and equatorial semirings are studied in [23].

Another classic question for the language of regular expressions, with or without multiplicities, is the study of complete axiom systems for them (see, e.g., [9,21,33]). Along this line of research, Bonnier-Rigny and Krob have offered a complete system of identities for one-letter regular expressions with multiplicities in the tropical semiring [6]. However, to the best of our knowledge, there has not been a systematic investigation of the equational theories of the different exotic semirings studied in the literature. This is the aim of this paper.

Our starting points are the results we obtained in [1,2]. In [1] we studied the equational theory of the max-plus algebra of the natural numbers $\mathbf{N}_{\vee} = (\mathbb{N}, \vee, +, 0)$, and proved that not only its equational theory is not finitely based, but, for every natural number n, the equations in at most n variables that hold in it do not form an equational basis. Another view of the non-existence of a finite basis for the variety generated by this algebra is offered in [2], where we showed that the collection of equations in two variables that hold in it has no finite equational axiomatization.

The algebra $\mathbf{N}_{\vee}$ is an example of a structure that we call in this paper *commutative idempotent weak semiring* (abbreviated henceforth to ciw-semiring). Since ciw-semirings underlie many of the exotic semirings studied in the literature, we begin our investigations in this paper by systematically generalizing the results from [1] to the structures $\mathbf{Z}_{\vee} = (\mathbb{Z}, \vee, +, 0)$ and $\mathbf{N}_{\wedge} = (\mathbb{N}, \wedge, +, 0)$. Our initial step in the study of the equational theories of these ciw-semirings is the geometric characterization of the (in)equations that hold in them (Propositions 17 and 19). These characterizations pave the way to explicit descriptions of the free algebras in the varieties $\mathscr{V}(\mathbf{Z}_{\vee})$ and $\mathscr{V}(\mathbf{N}_{\wedge})$ generated by $\mathbf{Z}_{\vee}$ and $\mathbf{N}_{\wedge}$, respectively, (Theorems 23 and 24) and yield finite axiomatizations of the varieties $\mathscr{V}(\mathbf{N}_{\vee})$ and $\mathscr{V}(\mathbf{N}_{\wedge})$ relative to $\mathscr{V}(\mathbf{Z}_{\vee})$ (Theorem 30). We then show that, like $\mathscr{V}(\mathbf{N}_{\vee})$, the varieties $\mathscr{V}(\mathbf{Z}_{\vee})$ and $\mathscr{V}(\mathbf{N}_{\wedge})$ are not finitely based. The non-finite axiomatizability of the variety $\mathscr{V}(\mathbf{Z}_{\vee})$ (Theorem 30) is a consequence of the similar result for $\mathscr{V}(\mathbf{N}_{\vee})$ and of its finite axiomatizability relative to $\mathscr{V}(\mathbf{Z}_{\vee})$.

The proof of the non-existence of a finite basis for the variety $\mathscr{V}(\mathbf{N}_{\wedge})$ (Theorem 31) is more challenging, and proceeds as follows. For each $n \geqslant 3$, we first isolate an equation $e_n^{\wedge}$ in n variables which holds in $\mathscr{V}(\mathbf{N}_{\wedge})$. We then prove that no finite collection of equations that hold in $\mathscr{V}(\mathbf{N}_{\wedge})$ can be used to deduce all of the equations of the form $e_n^{\wedge}$. The proof of this technical result is model-theoretic in nature. More precisely, for every natural number $n \geqslant 3$, we construct an algebra $\mathbf{B}_n$ satisfying all the equations in at most $n-1$ variables that hold in $\mathscr{V}(\mathbf{N}_{\wedge})$, but in which $e_n^{\wedge}$ fails. Hence, as for $\mathscr{V}(\mathbf{N}_{\vee})$, for every natural number n, the equations in at most n variables that hold in $\mathscr{V}(\mathbf{N}_{\wedge})$ do not form an equational basis for this variety. A similar strengthening of the non-finite axiomatizability result holds for the variety $\mathscr{V}(\mathbf{Z}_{\vee})$.

We then move on to study the equational theories of the exotic semirings presented in the literature that are obtained by adding bottom elements to the above ciw-semirings. More specifically, we examine the following semirings:

$$\mathbf{Z}_{\vee,-\infty} = (\mathbb{Z} \cup \{-\infty\}, \vee, +, -\infty, 0),$$
$$\mathbf{N}_{\vee,-\infty} = (\mathbb{N} \cup \{-\infty\}, \vee, +, -\infty, 0)$$

and

$$\mathbf{N}^-_{\vee,-\infty} = (\mathbb{N}^- \cup \{-\infty\}, \vee, +, -\infty, 0),$$

where $\mathbb{N}^-$ stands for the set of non-positive integers. Since $\mathbf{Z}_{\vee,-\infty}$ and $\mathbf{N}^-_{\vee,-\infty}$ are easily seen to be isomorphic to the semirings

$$\mathbf{Z}_{\wedge,\infty} = (\mathbb{Z} \cup \{\infty\}, \wedge, +, \infty, 0)$$

and

$$\mathbf{N}_{\wedge,\infty} = (\mathbb{N} \cup \{\infty\}, \wedge, +, \infty, 0),$$

respectively, the results that we obtain apply equally well to these algebras. (The semirings $\mathbf{Z}_{\wedge,\infty}$ and $\mathbf{N}_{\wedge,\infty}$ are usually referred to as the *equatorial semiring* [22] and the *tropical semiring* [35], respectively. The semiring $\mathbf{N}_{\vee,-\infty}$ is called the *polar semiring* in [24].)

Our study of the equational theories of these algebras will proceed as follows. First, we shall offer some general facts relating the equational theory of a ciw-semiring $\mathbf{A}$ to that of the free commutative idempotent semiring $\mathbf{A}_\perp$ it generates. In particular, in Section 4.1 we shall relate the non-finite axiomatizability of the variety $\mathscr{V}(\mathbf{A}_\perp)$ generated by $\mathbf{A}_\perp$ to the non-finite axiomatizability of the variety $\mathscr{V}(\mathbf{A})$ generated by $\mathbf{A}$. Then, in Section 4.2, we shall apply our general study to derive the facts that all of the tropical semirings studied in this paper have exponential time decidable, but non-finitely based equational theories. Our general results, together with those proven in Section 3, will also give geometric characterizations of the valid equations in the tropical semirings $\mathbf{Z}_{\vee,-\infty}$ and $\mathbf{N}^-_{\vee,-\infty}$, but not in $\mathbf{N}_{\vee,-\infty}$. The task of providing a geometric description of the valid equations for the semiring $\mathbf{N}_{\vee,-\infty}$ will be accomplished in Section 4.3, where we shall also show that $\mathscr{V}(\mathbf{N}_{\vee,-\infty})$ can be axiomatized over $\mathscr{V}(\mathbf{Z}_{\vee,-\infty})$ by a single equation.

Some semirings studied in the literature are obtained by adding a top element $\top$ to a ciw-semiring $\mathbf{A}$ in lieu of a bottom element. We examine the general relationships that exist between the equational theory of a ciw-semiring $\mathbf{A}$ and that of the semiring $\mathbf{A}_\top$ so generated in Section 5. This general theory, together with the previously obtained non-finite axiomatizability results, is then applied to show that several semirings with $\top$ are not finitely based either.

We conclude our investigations by examining some variations on the aforementioned semirings. These include, amongst others, structures whose carrier sets are the (non-negative) rational or real numbers (Section 6.1), semirings whose product operation is standard multiplication (Section 6.2), the semirings studied by Mascle and Leung in

[30,26,27], respectively, (Section 6.3), and a min-plus algebra based on the ordinals proposed by Mascle in [29] (Section 6.4). For all of these structures, we offer results to the effect that their equational theory is not finitely based, and has no axiomatization in a bounded number of variables.

Throughout the paper, we shall use standard notions and notations from universal algebra that can be found, e.g., in [8,13].

This paper collects, and improves upon, all of the results first announced without proof in [3,4]. In addition, the material in Sections 5 and 6 is new.

2. Preliminaries

We begin by introducing some notions that will be used in the technical developments to follow.

A *commutative idempotent weak semiring* (henceforth abbreviated to ciw-semiring) is an algebra $\mathbf{A} = (A, \vee, +, 0)$ such that $(A, \vee)$ is an idempotent commutative semigroup, i.e., a semilattice, $(A, +, 0)$ is a commutative monoid, and such that addition distributes over the $\vee$ operation. Thus, the following equations hold in $\mathbf{A}$:

$$x \vee (y \vee z) = (x \vee y) \vee z$$
$$x \vee y = y \vee x$$
$$x \vee x = x$$
$$x + (y + z) = (x + y) + z$$
$$x + y = y + x$$
$$x + 0 = x$$
$$x + (y \vee z) = (x + y) \vee (x + z).$$

A ciw-semiring $\mathbf{A}$ is *positive* if

$$x \vee 0 = x$$

holds in $\mathbf{A}$. It then follows that

$$x \vee (x + y) = x + y \tag{1}$$

also holds in $\mathbf{A}$. A homomorphism of ciw-semirings is a function which preserves the $\vee$ and $+$ operations and the constant 0.

A *commutative idempotent semiring*, or *ci-semiring* for short, is an algebra $(A, \vee, +, \perp, 0)$ such that $(A, \vee, +, 0)$ is a ciw-semiring which satisfies the equations

$$x \vee \perp = x$$

and

$$x + \perp = \perp.$$

A homomorphism of ci-semirings also preserves $\perp$.

Suppose that $\mathbf{A} = (A, \vee, +, 0)$ is a structure equipped with binary operations $\vee$ and $+$ and the constant 0. Assume that $\perp \notin A$ and let $A_\perp = A \cup \{\perp\}$. Extend the operations $\vee$ and $+$ given on A to $A_\perp$ by defining

$$a \vee \perp = \perp \vee a = a$$
$$a + \perp = \perp + a = \perp,$$

for all $a \in A_\perp$. We shall write $\mathbf{A}_\perp$ for the resulting algebra $(A_\perp, \vee, +, \perp, 0)$, and $\mathbf{A}(\perp)$ for the algebra $(A_\perp, \vee, +, 0)$ obtained by adding $\perp$ to the carrier set, but not to the signature.

Lemma 1. *For each ciw-semiring* $\mathbf{A}$, *the algebra* $\mathbf{A}_\perp$ *is a ci-semiring.*

Remark 2. In fact, $\mathbf{A}_\perp$ is the free ci-semiring generated by $\mathbf{A}$.

Let E_{ciw} denote the set of defining axioms of ciw-semirings, E_{ci} the set of axioms of ci-semirings, and E_{ciw}^+ the set of axioms of positive ciw-semirings. Note that E_{ciw} is included in both E_{ci} and E_{ciw}^+. Moreover, let $\mathscr{V}_{\mathrm{ciw}}$ denote the variety axiomatized by E_{ciw}, $\mathscr{V}_{\mathrm{ci}}$ the variety axiomatized by E_{ci} and $\mathscr{V}_{\mathrm{ciw}}^+$ the variety axiomatized by E_{ciw}^+. Thus, $\mathscr{V}_{\mathrm{ci}}$ is the variety of all ci-semirings and $\mathscr{V}_{\mathrm{ciw}}^+$ the variety of all positive ciw-semirings. Since $E_{\mathrm{ciw}} \subseteq E_{\mathrm{ci}}$ and $E_{\mathrm{ciw}} \subseteq E_{\mathrm{ciw}}^+$, it follows that $\mathscr{V}_{\mathrm{ciw}}$ includes both $\mathscr{V}_{\mathrm{ciw}}^+$ and the reduct of any algebra in $\mathscr{V}_{\mathrm{ci}}$ obtained by forgetting about the constant $\perp$.

Remark 3. If an equation $t = u$ can be derived from E_{ciw} or E_{ciw}^+, then the set of variables occurring in t coincides with the set of variables occurring in u. In light of axiom $x + \perp = \perp$, this does not hold true for the equations derivable from E_{ci}.

Notation 4. In the remainder of this paper, we shall use nx to denote the n-fold addition of x with itself, and we take advantage of the associativity and commutativity of the operations. By convention, nx stands for 0 when $n = 0$. In the same way, the empty sum is defined to be 0.

For each integer $n \geqslant 0$, we use $[n]$ to stand for the set $\{1, \ldots, n\}$, so that $[0]$ is another name for the empty set.

Finally, we sometimes write $t(x_1, \ldots, x_n)$ to emphasize that the variables occurring in the term t are amongst $x_1, \ldots, x_n$.

Lemma 5. *With respect to the axiom system* E_{ciw}, *every term* t *in the language of ciw-semirings, in the variables* $x_1, \ldots, x_n$, *may be rewritten in the form*

$$t = \bigvee_{i \in [k]} t_i$$

where $k > 0$, *each* t_i *is a "linear combination"*

$$t_i = \sum_{j \in [n]} c_{ij} x_j$$

of the variables $x_1, \ldots, x_n$, *and each* c_{ij} *is in* $\mathbb{N}$.

Terms of the form $\bigvee_{i \in [k]} t_i$, where each t_i ($i \in [k]$, $k \geqslant 0$) is a linear combination of variables, will be referred to as *simple terms*. When $k = 0$, the term $\bigvee_{i \in [k]} t_i$ is just $\perp$. (Note that $k = 0$ is only allowed for ci-semirings.)

Lemma 6. *With respect to the axiom system* E_{ci}, *every term* t *in the language of ci-semirings, in the variables* $x_1, \ldots, x_n$, *may be rewritten in the form*

$$t = \bigvee_{i \in [k]} t_i$$

where $k \geqslant 0$, *each* t_i *is a "linear combination"*

$$t_i = \sum_{j \in [n]} c_{ij} x_j$$

of the variables $x_1, \ldots, x_n$, *and each* c_{ij} *is in* $\mathbb{N}$.

For any commutative idempotent (weak) semiring $\mathbf{A}$ and $a, b \in A$, we write $a \leqslant b$ to mean $a \vee b = b$. In any such structure, the relation $\leqslant$ so defined is a partial order, and the $+$ and $\vee$ operations are monotonic with respect to it. Similarly, we say that an inequation $t \leqslant t'$ between terms t and t' holds in $\mathbf{A}$ if the equation $t \vee t' = t'$ holds. We shall write $\mathbf{A} \models t = t'$ (respectively, $\mathbf{A} \models t \leqslant t'$) if the equation $t = t'$ (resp., the inequation $t \leqslant t'$) holds in $\mathbf{A}$. (In that case, we say that $\mathbf{A}$ is a model of $t = t'$ or $t \leqslant t'$, respectively.) If $\mathscr{A}$ is a class of ciw-semirings, we write $\mathscr{A} \models t = t'$ (respectively, $\mathscr{A} \models t \leqslant t'$) if the equation $t = t'$ (resp., the inequation $t \leqslant t'$) holds in every $\mathbf{A} \in \mathscr{A}$. Note that, if $\mathbf{A}$ is in the variety $\mathscr{V}_{\mathrm{ciw}}^{+}$, then the inequation $0 \leqslant x$ holds in $\mathbf{A}$.

Definition 7. A *simple inequation* (sometimes referred to as *simple* $\vee$-*inequation* for the sake of clarity) in the variables $x_1, \ldots, x_n$ is of the form

$$t \leqslant \bigvee_{i \in [k]} t_i,$$

where $k > 0$, and t and the t_i ($i \in [k]$) are linear combinations of the variables $x_1, \ldots, x_n$. We say that the left-hand side of the above simple inequation contains the variable x_j, or that x_j appears on the left-hand side of the simple inequation, if the coefficient of x_j in t is non-zero. Similarly, we say that the right-hand side of the above inequation contains the variable x_j if for some $i \in [k]$, the coefficient of x_j in t_i is non-zero.

Note that, for every linear combination t over variables $x_1, \ldots, x_n$, the inequation $t \leqslant \perp$ is *not* a simple inequation.

Corollary 8. *With respect to the axiom system* E_{ciw}, *any equation in the language of ciw-semirings is equivalent to a finite set of simple inequations. Similarly, with respect to* E_{ci}, *any equation in the language of ci-semirings is equivalent to a finite set of simple inequations or to an inequation of the form* $x \leqslant \perp$ (*in which case the equation has, in conjunction with* E_{ci}, *only trivial models*).

Proof. Let $t = u$ be an equation in the language of ciw-semirings. By Lemma 5, using E_{ciw} we can rewrite t and u to simple terms $\bigvee_{i \in [k]} t_i$ and $\bigvee_{j \in [l]} u_j$ ($k, l > 0$), respectively. It is now immediate to see that the equation $t = u$ is equivalent to the family of simple inequations

$$\left\{ t_i \leqslant \bigvee_{j \in [l]} u_j, \ u_j \leqslant \bigvee_{i \in [k]} t_i : \ i \in [k], \ j \in [l] \right\}.$$

Assume now that $t = u$ is an equation in the language of ci-semirings. By Lemma 6, using E_{ci} we can rewrite t and u to simple terms $\bigvee_{i \in [k]} t_i$ and $\bigvee_{j \in [l]} u_j$ ($k, l \geqslant 0$), respectively. If k, l are both positive or both equal to zero, then the equation $t = u$ is equivalent to the finite set of simple inequations given above. (Note that, if k and l are both 0, then this set is empty, and the original equation is equivalent to $\bot = \bot$.) If $k = 0$ and $l > 0$, say, then the equation is equivalent, modulo E_{ci}, to $x \leqslant \bot$. Indeed, in this case, the set of simple inequations given above becomes

$$\{ u_j \leqslant \bot : \ j \in [l] \}.$$

Substituting 0 for every variable occurring in u_1, we derive, using E_{ci}, that

$$0 \leqslant \bot.$$

Thus any ci-semiring satisfying this equation is positive. By adding x to both sides of the above inequation, we can infer, again using E_{ci}, the inequation

$$x \leqslant \bot,$$

which was to be shown. □

Notation 9. Henceforth in this study, we shall often abbreviate a simple inequation

$$\sum_{j \in [n]} d_j x_j \leqslant \bigvee_{i \in [k]} \sum_{j \in [n]} c_{ij} x_j,$$

in the variables $\bar{x} = (x_1, \ldots, x_n)$, as $\bar{d} \leqslant \{ \bar{c}_1, \ldots, \bar{c}_k \}$, where $\bar{d} = (d_1, \ldots, d_n)$ and $\bar{c}_i = (c_{i1}, \ldots, c_{in})$, for $i \in [k]$. We shall sometimes refer to these inequations as *simple $\vee$-inequations*. Moreover, we shall often use $\bar{d} \cdot \bar{x}$ as a shorthand for

$$d_1 x_1 + \cdots + d_n x_n.$$

The $\cdot$ will often be omitted from $\bar{d} \cdot \bar{x}$.

In the main body of the paper, we shall also study some ciw-semirings that, like the structure $\mathbf{N}_\wedge = (\mathbb{N}, \wedge, +, 0)$, have the minimum operation in lieu of maximum. The preliminary results that we have developed in this section apply equally well to these structures. In particular, the axioms for ciw-semirings dealing with $\vee$ become the standard ones describing the obvious identities for the minimum operation, and its

interplay with $+$, i.e.,

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$
$$x \wedge y = y \wedge x$$
$$x \wedge x = x$$
$$x + (y \wedge z) = (x + y) \wedge (x + z).$$

Note that, for ciw-semirings of the form $(A, \wedge, +, 0)$, the partial order $\geqslant$ is defined by $b \geqslant a$ iff $a \wedge b = a$. In Definition 7, we introduced the notion of simple $\vee$-inequation. Dually, we say that a *simple $\wedge$-inequation* in the variables $\bar{x} = (x_1, \ldots, x_n)$ is an inequation of the form

$$\bar{d} \cdot \bar{x} \geqslant \bigwedge_{i \in [k]} \bar{c}_i \cdot \bar{x},$$

where $\bar{d}$ and the $\bar{c}_i$ ($i \in [k]$) are vectors in $\mathbb{N}^n$. We shall often write

$$\bar{d} \geqslant \{\bar{c}_1, \ldots, \bar{c}_k\}$$

as a shorthand for this inequation.

3. Min-max-plus weak semirings

Our aim in this section will be to study the equational theories of the ciw-semirings that underlie most of the tropical semirings studied in the literature. More specifically, we shall study the following ciw-semirings:

$$\mathbf{Z}_\vee = (\mathbb{Z}, \vee, +, 0),$$
$$\mathbf{N}_\vee = (\mathbb{N}, \vee, +, 0),$$
$$\mathbf{N}_\wedge = (\mathbb{N}, \wedge, +, 0),$$

equipped with the usual addition operation $+$, constant 0 and one of the operations $\vee$ (for the maximum of two numbers) and $\wedge$ (for the minimum of two numbers), i.e.,

$$x \vee y = \max\{x, y\}$$

and

$$x \wedge y = \min\{x, y\}.$$

We shall sometimes use the fact that $\mathbf{Z}_\vee$ and $\mathbf{N}_\wedge$ are isomorphic to the ciw-semirings

$$\mathbf{Z}_\wedge = (\mathbb{Z}, \wedge, +, 0)$$

and

$$\mathbf{N}_\vee^- = (\mathbb{N}^-, \vee, +, 0),$$

respectively, where $\mathbb{N}^-$ stands for the set of non-positive integers.

Our study of the equational theories of these algebras will be based on the following uniform pattern. First, we offer geometric characterizations of the simple inequations that hold in these ciw-semirings (Section 3.2). These characterizations pave the way to concrete descriptions of the free algebras in the varieties generated by the algebras we study, and yield relative axiomatization and decidability results (Section 3.3). Finally we show that none of the ciw-semirings we study is finitely based (Section 3.4). All of these technical results rely on a study of properties of convex sets, filters and ideals in $\mathbb{Z}^n$ and $\mathbb{N}^n$ presented in the following section.

3.1. Convex sets, filters and ideals

Suppose that $\bar{v}_1, \ldots, \bar{v}_k$ $(k > 0)$ are vectors in $\mathbb{Z}^n$ or, more generally, in $\mathbb{R}^n$. A *convex linear combination* of the $\bar{v}_i$ $(i \in [k])$ is any vector $\bar{v} \in \mathbb{R}^n$ which can be written as

$$\bar{v} = \lambda_1 \bar{v}_1 + \cdots + \lambda_k \bar{v}_k,$$

where $\lambda_i \geqslant 0$, $i \in [k]$, are real numbers with $\sum_{i=1}^{k} \lambda_i = 1$.

Definition 10. Suppose that U is any subset of $\mathbb{Z}^n$. We call U a *convex set* if every vector in $\mathbb{Z}^n$ that is a convex linear combination of vectors in U is also contained in U.

Suppose that $U \subseteq \mathbb{N}^n$. We call U an (*order*) *ideal* if for all $\bar{u}, \bar{v}$ in $\mathbb{N}^n$, whenever $\bar{u} \leqslant \bar{v}$, with respect to the pointwise order over $\mathbb{N}^n$, and $\bar{v} \in U$ then $\bar{u} \in U$. Moreover, we call U a *filter*, if for all $\bar{u}$ and $\bar{v}$ as above, if $\bar{u} \in U$ and $\bar{u} \leqslant \bar{v}$ then $\bar{v} \in U$. A *convex ideal* (respectively, *convex filter*) in $\mathbb{N}^n$ is any ideal (resp., filter) which is a convex set.

Note that order ideals and filters are sometimes referred to as lower and upper sets, respectively.

The following fact is easy to prove:

Proposition 11. *The intersection of any number of convex sets in $\mathbb{Z}^n$ is convex. Moreover, the intersection of any number of convex ideals (convex filters) in $\mathbb{N}^n$ is a convex ideal (convex filter, respectively).*

Thus each set $U \subseteq \mathbb{Z}^n$ is contained in a smallest convex set $[U]$ which is the intersection of all convex subsets of $\mathbb{Z}^n$ containing U. We call $[U]$ the convex set generated by U, or the *convex hull* of U. When $\bar{u} \in \mathbb{Z}^n$, below we shall sometimes write $[\bar{u}]$ for $[\{\bar{u}\}]$. Observe that $\{\bar{u}\}$ is convex, so that $[\bar{u}] = [\{\bar{u}\}] = \{\bar{u}\}$.

Note that we have $[U] \subseteq \mathbb{N}^n$ whenever $U \subseteq \mathbb{N}^n$.

The following proposition provides an explicit description of $[U]$, for any given $U \subseteq \mathbb{Z}^n$. It relies on the well-known fact that any convex linear combination of convex linear combinations of some vectors $\bar{v}_1, \ldots, \bar{v}_k \in \mathbb{Z}^n$ is itself a convex linear combination of $\bar{v}_1, \ldots, \bar{v}_k$.

Proposition 12. *Suppose that $U \subseteq \mathbb{Z}^n$ and $\bar{v} \in \mathbb{Z}^n$. We have $\bar{v} \in [U]$ iff $\bar{v}$ is a convex linear combination of some non-zero number of vectors in U.*

Suppose now that $U \subseteq \mathbb{N}^n$. By Proposition 11, there are a smallest convex ideal $\mathsf{ci}(U)$ and a smallest convex filter $\mathsf{cf}(U)$ in $\mathbb{N}^n$ containing U. We call $\mathsf{ci}(U)$ and $\mathsf{cf}(U)$ the convex ideal and the convex filter generated by U, respectively.

For each set $U \subseteq \mathbb{R}^n$, define the ideal $(U]$ generated by U thus

$$(U] = \{\bar{d} \in \mathbb{N}^n \colon \exists \bar{c} \in U.\ \bar{d} \leqslant \bar{c}\}.$$

Similarly, the filter $[U)$ generated by U is defined as

$$[U) = \{\bar{d} \in \mathbb{N}^n \colon \exists \bar{c} \in U.\ \bar{d} \geqslant \bar{c}\}.$$

The following proposition will be useful in what follows. In its proof, and throughout this study, we shall use $\bar{p}_i$ ($i \in [n]$) to denote the ith unit vector in $\mathbb{R}^n$, i.e., the vector whose only non-zero component is a 1 in the ith position; $\bar{0}$ will denote the vector in $\mathbb{R}^n$ whose entries are all zero. When $\bar{u} \in \mathbb{N}^n$, below we shall sometimes write $(\bar{u}]$ and $[\bar{u})$ for $(\{\bar{u}\}]$ and $[\{\bar{u}\})$, respectively.

In the following proposition, and in its proof, we write $[U]_{\mathbb{R}^n}$ for the convex hull in $\mathbb{R}^n$ of a set U included in $\mathbb{N}^n$.

Proposition 13. *Suppose that $U \subseteq \mathbb{N}^n$. Then*:
(1) $\mathsf{ci}(U) = [(U]] = ([U]_{\mathbb{R}^n}]$ *and*
(2) $\mathsf{cf}(U) = [[U)] = [[U]_{\mathbb{R}^n})$.

Proof. We prove the two claims separately.
(1) Note, first of all, that

$$[(U]] \subseteq \mathsf{ci}(U)$$

holds, as U is included in $\mathsf{ci}(U)$, and $\mathsf{ci}(U)$ is a convex ideal. Moreover, any convex linear combination of vectors in $([U]_{\mathbb{R}^n}]$ is below a convex linear combination of elements of U. It follows that $([U]_{\mathbb{R}^n}]$ is convex, and thus

$$\mathsf{ci}(U) \subseteq ([U]_{\mathbb{R}^n}].$$

To complete the proof, it is therefore sufficient to show that $([U]_{\mathbb{R}^n}]$ is included in $[(U]]$. This is obvious if $n = 1$. In order to proceed with the proof of this claim, suppose now that

$$\bar{d} \leqslant \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k$$

for some $k > 0$, $\bar{d} \in \mathbb{N}^n$, $\bar{c}_1, \ldots, \bar{c}_k \in U$ and $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i \in [k]} \lambda_i = 1$. It suffices to show that $\bar{d}$ is in the convex hull of the set $(V]$, where $V = \{\bar{c}_1, \ldots, \bar{c}_k\}$. We shall prove this by induction on

$$r = n + w,$$

where w is the sum of all of the entries of the vectors $\bar{c}_1, \ldots, \bar{c}_k$.

The base case is when $r = 1$. Then $n = 1$ and we are done.

For the inductive step, suppose that $r > 1$. We proceed with the proof by distinguishing two cases.

Case 1: If there exists some $j \in [n]$ with $d_j = 0$, then since we aim at showing that $\bar{d} \in [(V)]$, without loss of generality, we may assume that, for this j, the jth component of each $\bar{c}_i$ is 0—i.e., $c_{ij} = 0$ for all $i \in [k]$. We can then remove the jth components of all the vectors to obtain $\bar{d}'$ and $\bar{c}'_1, \ldots, \bar{c}'_k$ of dimension $n-1$ with $\bar{d}' \leqslant \lambda_1 \bar{c}'_1 + \cdots + \lambda_k \bar{c}'_k$. Let $W = \{\bar{c}'_1, \ldots, \bar{c}'_k\}$. By induction, $\bar{d}'$ is in the convex hull of $(W]$, so that $\bar{d}$ is in the convex hull of $(V]$. The case that $n = 1$ is trivial.

Case 2: If the previous case does not apply, then $d_j > 0$ for all $j \in [n]$. In this case, for each $j \in [n]$ there is some $i_j \in [k]$ with $c_{i_j j} \geqslant 1$. Let $\bar{e} = \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k$. If for some $j \in [n]$

$$\bar{d} \leqslant \lambda_1 \bar{c}_1 + \cdots + \lambda_{i_j-1} \bar{c}_{i_j-1} + \lambda_{i_j}(\bar{c}_{i_j} - \bar{p}_j) + \lambda_{i_j+1} \bar{c}_{i_j+1} + \cdots + \lambda_k \bar{c}_k$$
$$= \bar{e} - \lambda_{i_j} \bar{p}_j,$$

where $\bar{p}_j$ denotes the jth unit vector in $\mathbb{N}^n$, then, by induction, $\bar{d}$ is contained in the convex hull of $(W]$, where W is the set

$$\{\bar{c}_1, \ldots, \bar{c}_{i_j-1}, \bar{c}_{i_j} - \bar{p}_j, \bar{c}_{i_j+1}, \ldots \bar{c}_k\} \subseteq (V].$$

It follows that $\bar{d}$ is in the convex hull of $(V]$. Otherwise, we have that

$$e_j - \lambda_{i_j} \leqslant d_j \leqslant e_j,$$

for all $j \in [n]$. This means that $\bar{d}$ is inside the n-dimensional cube determined by the vectors

$$\bar{v}_K = \bar{e} - \sum_{j \in K} \lambda_{i_j} \bar{p}_j,$$

where K ranges over all subsets of $[n]$. Since these vectors $\bar{v}_K$ are all in the convex hull in $\mathbb{R}^n$ of $(V]$, it follows that $\bar{d}$ belongs to the convex hull of $(V]$, which was to be shown.

(2) As in the proof of the first claim, we can argue that

$$[[U)] \subseteq \mathsf{cf}(U) \subseteq [[U]_{\mathbb{R}^n}).$$

To complete the proof, we show that $[[U]_{\mathbb{R}^n})$ is included in $[[U)]$.

To this end, let $\bar{d} \in [[U]_{\mathbb{R}^n})$. This means that there is a convex linear combination $\bar{c} = \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k$ with $\bar{d} \geqslant \bar{c}$ and $\{\bar{c}_1, \ldots, \bar{c}_k\} \subseteq U$. Let $\bar{e}$ be a vector in $\mathbb{N}^n$ which is maximal with respect to the pointwise order such that

$$\bar{d} \geqslant \lambda_1(\bar{c}_1 + \bar{e}) + \cdots + \lambda_k \bar{c}_k.$$

Then $\bar{d} \in [[\{\bar{c}_1 + \bar{e}, \ldots, \bar{c}_k\}]_{\mathbb{R}^n})$. As

$$[\{\bar{c}_1 + \bar{e}, \ldots, \bar{c}_k\}) \subseteq [\{\bar{c}_1, \ldots, \bar{c}_k\}) \subseteq [U),$$

to prove the claim it is sufficient to show that $\bar{d} \in [[\{\bar{c}_1 + \bar{e}, \ldots, \bar{c}_k\})]]$. Therefore, without loss of generality, we may assume that $\bar{e} = \bar{0}$. Now we proceed with the proof as follows. Let $j \in [n]$. Since $\bar{d}$ is not greater than or equal to a convex linear combination of $\bar{c}_1 + \bar{p}_j, \bar{c}_2, \ldots, \bar{c}_k$, where $\bar{p}_j$ denotes the jth unit vector in $\mathbb{N}^n$, we have

$$\bar{d} \not\geqslant \bar{c} + \lambda_1 \bar{p}_j = \lambda_1(\bar{c}_1 + \bar{p}_j) + \lambda_2 \bar{c}_2 + \cdots + \lambda_k \bar{c}_k.$$

Thus, for d_j, viz. the jth component of $\bar{d}$, it holds that

$$\sum_{i=1}^{k} \lambda_j c_{ij} \leqslant d_j < \lambda_1 + \sum_{i=1}^{k} \lambda_j c_{ij}.$$

As before, for any $i \in [k]$, c_{ij} denotes the jth component of $\bar{c}_i$. For each $K \subseteq [n]$, define

$$\bar{c}_K = \bar{c} + \sum_{j \in K} \lambda_1 \bar{p}_j = \lambda_1 \left(\bar{c}_1 + \sum_{j \in K} \bar{p}_j \right) + \lambda_2 \bar{c}_2 + \cdots + \lambda_k \bar{c}_k,$$

so that $\bar{c}_\emptyset = \bar{c}$ and each $\bar{c}_K$ is a convex linear combination of vectors in $[\{\bar{c}_1, \ldots, \bar{c}_k\})$. The vectors $\bar{c}_K$ determine an n-dimensional cube, and, by the above reasoning, $\bar{d}$ is in this cube. Therefore, $\bar{d}$ is a convex linear combination of the $\bar{c}_K$. In conclusion, $\bar{d}$ is a convex linear combination of vectors in $[\{\bar{c}_1, \ldots, \bar{c}_k\}) \subseteq [U)$, which was to be shown. $\square$

Remark 14. It is interesting to note that the equalities $\mathsf{ci}(U) = ([U]]$ and $\mathsf{cf}(U) = [[U])$ fail. In fact, the ideal and the filter generated by a convex set are, in general, not convex. Consider, for example, the case $n = 2$ and $U = \{(0, 2), (3, 0)\}$. Then U is convex (cf. Definition 10) since no non-trivial convex combination of the two vectors in U yields a point in $\mathbb{N}^2$. Hence $([U]] = (U]$ which contains $(0, 2)$ and $(2, 0)$. However, the convex combination

$$\frac{1}{2}(0, 2) + \frac{1}{2}(2, 0) = (1, 1)$$

does not belong to $(U]$. Similarly, the filter $[[U]) = [U)$ contains $(1, 2)$ and $(3, 0)$, but not

$$\frac{1}{2}(1, 2) + \frac{1}{2}(3, 0) = (2, 1).$$

Thus, it is not convex and cannot be equal to $\mathsf{cf}(U)$ or to $[[U])$.

Henceforth, we shall use $[(U]]$ and $[[U)]$ to denote the convex ideal and the convex filter generated by U, respectively. This notation is justified by the above proposition.

Corollary 15. *A set $U \subseteq \mathbb{N}^n$ is a convex ideal (respectively, convex filter) iff for every $\bar{d} \in \mathbb{N}^n$, whenever $\bar{d} \leqslant \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k$ (resp., $\bar{d} \geqslant \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k$) where $k > 0$, $\lambda_i \geqslant 0$, $\bar{c}_i \in U$ ($i \in [k]$) and $\sum_{i=1}^{k} \lambda_i = 1$, it follows that $\bar{d} \in U$.*

Remark 16. Each filter in $\mathbb{N}^n$, be it convex or not, is finitely generated. Indeed, the set of minimal elements, with respect to the pointwise partial order, in any filter in $\mathbb{N}^n$ is finite, since any antichain in $\mathbb{N}^n$ is finite. Moreover, it is easy to see that it generates the filter, and is, in fact, its unique minimal set of generators with respect to set inclusion.

Each finite convex set in $\mathbb{Z}^n$ has a unique minimal generating set. Indeed, the collection of vertices of such a set is its only minimal set of generators [39, Section 3.1]. (We recall that a vertex of a convex set is a point that cannot be expressed as a convex linear combination of other points in the set—see, e.g., [39, p. 80].) Each finite (convex) ideal in $\mathbb{N}^n$ also has a unique minimal generating set.

3.2. Characterization of valid inequations

Recall that a simple $\vee$-inequation in the variables $\bar{x} = (x_1, \ldots, x_n)$ is an inequation of the form

$$\bar{d}\bar{x} \leqslant \bar{c}_1 \bar{x} \vee \cdots \vee \bar{c}_k \bar{x}, \tag{2}$$

where $k > 0$, and $\bar{d}, \bar{c}_1, \ldots, \bar{c}_k \in \mathbb{N}^n$. Similarly, a simple $\wedge$-inequation is of the form

$$\bar{d}\bar{x} \geqslant \bar{c}_1 \bar{x} \wedge \cdots \wedge \bar{c}_k \bar{x}, \tag{3}$$

where k, $\bar{d}$, and $\bar{c}_i$ $(i \in [k])$ are as above. We recall that (2) holds in a ciw-semiring $\mathbf{A}_\vee = (A, \vee, +, 0)$ if the equation

$$\bar{d}\bar{x} \vee \bar{c}_1 \bar{x} \vee \cdots \vee \bar{c}_k \bar{x} = \bar{c}_1 \bar{x} \vee \cdots \vee \bar{c}_k \bar{x}$$

does, i.e., when for all $\bar{v} \in A^n$,

$$\bar{d}\bar{v} \leqslant \bar{c}_1 \bar{v} \vee \cdots \vee \bar{c}_k \bar{v}.$$

Similarly, we say that (3) holds in a ciw-semiring $\mathbf{A}_\wedge = (A, \wedge, +, 0)$ if the equation

$$\bar{d}\bar{x} \wedge \bar{c}_1 \bar{x} \wedge \cdots \wedge \bar{c}_k \bar{x} = \bar{c}_1 \bar{x} \wedge \cdots \wedge \bar{c}_k \bar{x}$$

does. Let U denote the set $\{\bar{c}_1, \ldots, \bar{c}_k\}$. We recall that we shall sometimes abbreviate (2) as $\bar{d} \leqslant U$ and (3) as $\bar{d} \geqslant U$. For some structures, such as $\mathbf{Z}_\vee$, it also makes sense to define when a simple inequation $\bar{d} \leqslant U$ holds in $\mathbf{Z}_\vee$, where $\bar{d} \in \mathbb{Z}^n$ and $U = \{\bar{c}_1, \ldots, \bar{c}_k\}$ is a finite non-empty set of vectors in $\mathbb{Z}^n$.

We now proceed to characterize the collection of simple inequations (possibly with negative coefficients) that hold in the algebra $\mathbf{Z}_\vee$ (and, thus, in its isomorphic version $\mathbf{Z}_\wedge$).

Proposition 17. *Suppose that $\bar{d} \in \mathbb{Z}^n$ and $U = \{\bar{c}_1, \ldots, \bar{c}_k\}$ is a non-empty, finite set of vectors in $\mathbb{Z}^n$. Then the simple inequation $\bar{d} \leqslant U$ holds in $\mathbf{Z}_\vee$ iff $\bar{d}$ belongs to the set $[U]$.*

Proof. It is sufficient to prove this claim when $\bar{d} = \bar{0}$, for otherwise we can replace $\bar{d}$ by $\bar{0}$, the vector in $\mathbb{Z}^n$ whose components are all 0, and U by $U - \bar{d} = \{\bar{u} - \bar{d} : \bar{u} \in U\}$, respectively.

Suppose that $\bar{0} \in [U]$, i.e., that there exist real numbers $\lambda_1, \ldots, \lambda_k \geqslant 0$ with $\sum_{i=1}^{k} \lambda_i = 1$ such that

$$\bar{0} = \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k.$$

Thus, $0 = \lambda_1 \bar{c}_1 \bar{u} + \cdots + \lambda_k \bar{c}_k \bar{u}$ for all $\bar{u} \in \mathbb{Z}^n$. Since the λ_i are non-negative and at least one of them is non-zero, this is possible only if for each $\bar{u} \in \mathbb{Z}^n$ there exists some $i_0 \in [k]$ with $0 \leqslant \bar{c}_{i_0} \bar{u}$. It thus follows that $\bar{0} \leqslant U$ holds in $\mathbf{Z}_\vee$.

To prove the other direction, suppose that $\bar{0} \notin [U]$. We proceed to prove that $\bar{0} \leqslant U$ does not hold in $\mathbf{Z}_\vee$. To this end, we shall first exhibit a vector $\bar{u} \in \mathbb{R}^n$ such that $\bar{c}_i \bar{u} < 0$, for all $i \in [k]$, i.e., such that for every $i \in [k]$, the vectors $\bar{c}_i$ and $\bar{v} = -\bar{u}$ make an acute angle. But such a $\bar{u}$ is easy to find: let $\bar{u}$ be a vector in the convex hull in $\mathbb{R}^n$ of U whose endpoint is closest to the origin. (This exists, since the convex hull in $\mathbb{R}^n$ of U is a closed set.)

Let S denote the hyperplane passing through $\bar{u}$ and perpendicular to it. Let H stand for the halfspace determined by S which contains $\bar{0}$. If $\bar{w}$ is any point in the convex hull in $\mathbb{R}^n$ of U contained in H other than $\bar{u}$, then the line segment determined by $\bar{w}$ and $\bar{u}$ would contain a point in the convex hull in $\mathbb{R}^n$ of U closer to $\bar{0}$ than $\bar{u}$, contradicting our assumption about the vector $\bar{u}$. Hence the vectors $\bar{c}_i$ ($i \in [k]$) and $\bar{u}$ make an obtuse angle. It thus follows that the vectors $\bar{c}_i$ ($i \in [k]$) and $\bar{v}$ make an acute angle, as claimed.

Next we note that, for all $i \in [k]$, the function $\bar{x} \mapsto \bar{c}_i \cdot \bar{x}$ is continuous. Therefore, for each such i, there is a positive real number ε_i such that $\bar{c}_i \cdot \bar{x} < 0$ whenever $|\bar{u} - \bar{x}| < \varepsilon_i$ (where we use $|\bar{u} - \bar{x}|$ to denote the length of the vector $\bar{u} - \bar{x}$). Now take ε to be smallest amongst the ε_i ($i \in [k]$). Then, for all $i \in [k]$, it holds that $\bar{c}_i \cdot \bar{x} < 0$ whenever $|\bar{u} - \bar{x}| < \varepsilon$. In particular there must be a vector $\bar{x}$ with rational coefficients with this property. From this we derive easily that there must be a $\bar{w} \in \mathbb{Z}^n$ with $\bar{c}_i \cdot \bar{w} < 0$ for all $i \in [k]$. This shows that $\bar{0} \leqslant U$ does not hold in $\mathbf{Z}_\vee$, which was to be shown. $\quad\square$

Our order of business now will be to offer characterizations of the collections of simple inequations that hold in the algebras $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$. The following result connects the simple inequations that hold in these algebras, and will be useful to this effect.

Lemma 18. *For any $\bar{d}, \bar{c}_1, \ldots, \bar{c}_k$ in $\mathbb{N}^n$, where $k > 0$,*

$$\bar{d} \leqslant \{\bar{c}_1, \ldots, \bar{c}_k\} \tag{4}$$

holds in $\mathbf{N}_\vee$ iff

$$\bar{e} - \bar{d} \geqslant \{\bar{e} - \bar{c}_1, \ldots, \bar{e} - \bar{c}_k\} \tag{5}$$

holds in $\mathbf{N}_\wedge$, where for each $i \in [n]$, the ith component of $\bar{e} \in \mathbb{N}^n$ is the maximum of the ith components of $\bar{d}$ and the $\bar{c}_j$ ($j \in [k]$). In the same way,

$$\bar{d} \geqslant \{\bar{c}_1, \ldots, \bar{c}_k\}$$

holds in $\mathbf{N}_\wedge$ *iff*

$$\bar{e} - \bar{d} \leqslant \{\bar{e} - \bar{c}_1, \ldots, \bar{e} - \bar{c}_k\}$$

holds in $\mathbf{N}_\vee$, *where* $\bar{e}$ *is defined as above.*

Proof. We only prove the first statement of the lemma. Eq. (4) holds in $\mathbf{N}_\vee$ iff

$$\forall \bar{x} \in \mathbb{N}^n. \ \bar{d}\bar{x} \leqslant \bar{c}_1\bar{x} \vee \cdots \vee \bar{c}_k\bar{x}$$
$$\Leftrightarrow \forall \bar{x} \in \mathbb{N}^n \exists j \in [k]. \ \bar{d}\bar{x} \leqslant \bar{c}_j\bar{x}$$
$$\Leftrightarrow \forall \bar{x} \in \mathbb{N}^n \exists j \in [k]. \ (\bar{e} - \bar{d})\bar{x} \geqslant (\bar{e} - \bar{c}_j)\bar{x}$$
$$\Leftrightarrow \forall \bar{x} \in \mathbb{N}^n. \ (\bar{e} - \bar{d})\bar{x} \geqslant (\bar{e} - \bar{c}_1)\bar{x} \wedge \cdots \wedge (\bar{e} - \bar{c}_k)\bar{x},$$

viz. iff (5) holds in $\mathbf{N}_\wedge$. □

The above lemma expresses a "duality" between the equational theories of $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$. Note, however, that the equational theory of $\mathbf{N}_\wedge$ is *not* the formal dual of the theory of $\mathbf{N}_\vee$, since the equations

$$x \vee 0 = x$$

and

$$x \wedge 0 = 0$$

are not formal duals of each other.

Using Lemma 18 and results from [1], we are now in a position to offer the promised characterizations of the valid simple inequations in $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$.

Proposition 19. *Suppose that* $\bar{d} \in \mathbb{N}^n$ *and* U *is a finite non-empty set of vectors in* $\mathbb{N}^n$.
(1) *The simple inequation* $\bar{d} \leqslant U$ *holds in* $\mathbf{N}_\vee$ *iff* $\bar{d}$ *belongs to the set* $[(U]]$.
(2) *The simple inequation* $\bar{d} \geqslant U$ *holds in* $\mathbf{N}_\wedge$ *iff* $\bar{d}$ *belongs to the set* $[[U)]$.

Proof. The first claim is proved in [1]. The second follows from the first and Lemma 18. Let $U = \{\bar{c}_1, \ldots, \bar{c}_k\}$, say. Let $\bar{e} \in \mathbb{N}^n$ denote the vector whose jth component is the maximum of the jth components of $\bar{d}$ and the $\bar{c}_i$, for each $i \in [k]$. We know that $\bar{d} \geqslant \{\bar{c}_1, \ldots, \bar{c}_k\}$ holds in $\mathbf{N}_\wedge$ iff $\bar{e} - \bar{d} \leqslant \{\bar{e} - \bar{c}_1, \ldots, \bar{e} - \bar{c}_k\}$ holds in $\mathbf{N}_\vee$. But by the first claim in the lemma this holds iff there exist real numbers $\lambda_j \geqslant 0$, $j \in [k]$, with $\sum_{j=1}^k \lambda_j = 1$ and

$$\bar{e} - \bar{d} \leqslant \lambda_1(\bar{e} - \bar{c}_1) + \cdots + \lambda_k(\bar{e} - \bar{c}_k),$$

i.e., when

$$\bar{d} \geqslant \lambda_1\bar{c}_1 + \cdots + \lambda_k\bar{c}_k,$$

which was to be shown. □

As a corollary of Propositions 17 and 19, we obtain decidability results for the equational theories of the algebras $\mathbf{Z}_\vee, \mathbf{N}_\vee$ and $\mathbf{N}_\wedge$.

Corollary 20. *There exists an exponential time algorithm to decide whether an equation holds in the structures* $\mathbf{Z}_\vee, \mathbf{N}_\vee$ *and* $\mathbf{N}_\wedge$. *Moreover, it is decidable in polynomial time whether a simple inequation holds in these structures.*

Proof. The problem of deciding whether an equation holds in $\mathbf{Z}_\vee$, $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$, can be reduced to deciding whether a finite set of simple inequations holds (Corollary 8). The obvious reduction may result in a number of simple inequations that is exponential in the number of variables, and where each simple inequation has size that is linear in that of the original equation. However, the validity of a simple inequation can be tested in polynomial time by using linear programming (see, e.g., [34]). The interested reader is referred to [1] for more information. $\square$

Remark 21. The decidability of the equational theories of the structures $\mathbf{Z}_\vee$, $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$ also follows from well-known results in logic on the decidability of Presburger arithmetic—the first-order theory of addition on the natural numbers—and related theories. For example, it is well known (see, e.g., [31, Chap. 13] for a classic presentation) that the first-order theory with equality of the structure

$$(\mathbb{Z}, +, <, 0, 1, -)$$

is decidable, and the validity of any simple inequation in $\mathbf{Z}_\vee$ can easily be reduced to the validity of a first-order sentence over the language of the above structure.

It is interesting to compare the above result on the complexity of the equational theory of $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$ with the classic results by Fischer and Rabin [11] on the complexity of the first-order theory of the real numbers under addition, and of Presburger arithmetic. There is a fixed constant $c > 0$ such that for every (non-deterministic) decision procedure for determining the truth of sentences of real addition and for all sufficiently large n, there is a sentence of length n for which the decision procedure runs for more than 2^{cn} steps. In the case of Presburger arithmetic, the corresponding lower bound is $2^{2^{cn}}$. The lower bound $2^{2^{cn}}$ applies *mutatis mutandis* to the first-order theory of the algebras $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$.

3.3. Free algebras and relative axiomatizations

Let $C(\mathbb{N}^n)$, $CI(\mathbb{N}^n)$ and $CF(\mathbb{N}^n)$ denote the sets of all finite non-empty convex sets, finite non-empty convex ideals, and non-empty convex filters in $\mathbb{N}^n$, respectively. We turn each of these sets into a ciw-semiring. Suppose that $U, V \in C(\mathbb{N}^n)$. First of all, recall that the complex sum of U and V, notation $U \oplus V$, is defined thus:

$$U \oplus V = \{\bar{u} + \bar{v} : \ \bar{u} \in U, \ \bar{v} \in V\}.$$

We define

$$U \vee V = [U \cup V],$$
$$U + V = [U \oplus V],$$
$$0 = [\bar{0}] = \{\bar{0}\}.$$

We define the operations in $CI(\mathbb{N}^n)$ and $CF(\mathbb{N}^n)$ in a similar fashion. Suppose that $U, V \in CI(\mathbb{N}^n)$ and $U', V' \in CF(\mathbb{N}^n)$. We set

$$U \vee V = [(U \cup V)],$$
$$U + V = [(U \oplus V)],$$
$$U' \wedge V' = [[U' \cup V')],$$
$$U' + V' = [[U' \oplus V')].$$

Moreover, we define $0 = (\bar{0}] = \{\bar{0}\}$ in $CI(\mathbb{N}^n)$, and $0 = [\bar{0}) = \mathbb{N}^n$ in $CF(\mathbb{N}^n)$.

Proposition 22. *Each of the structures*

$$C(\mathbb{N}^n) = (C(\mathbb{N}^n), \vee, +, 0),$$
$$CI(\mathbb{N}^n) = (CI(\mathbb{N}^n), \vee, +, 0)$$

and

$$CF(\mathbb{N}^n) = (CF(\mathbb{N}^n), \wedge, +, 0)$$

is a ciw-semiring. In addition, $CI(\mathbb{N}^n)$ *satisfies the equation*

$$x \vee 0 = x \tag{6}$$

and $CF(\mathbb{N}^n)$ *the equation*

$$x \wedge 0 = 0. \tag{7}$$

Proof. The only non-trivial cases of the proof are the associativity for $+$ and $\vee$ together with the distributivity law. We only give the details of the proofs for the structure $C(\mathbb{N}^n)$ as similar arguments apply for the remaining ones. The proof is based on the following observations.

Let $A, B, C \subseteq \mathbb{N}^n$. It can easily be checked that

$$[A] \cup B \subseteq [A \cup B] \subseteq [[A] \cup B]$$

and therefore that

$$[[A] \cup B] = [A \cup B]. \tag{8}$$

Similarly it is not difficult to see that

$$[A] \oplus B \subseteq [A \oplus B] \subseteq [[A] \oplus B]$$

and hence that

$$[[A] \oplus B] = [A \oplus B]. \tag{9}$$

Towards proving the associativity law for $+$, assume that $U, V, W \in C(\mathbb{N}^n)$. By (9) and the associativity and commutativity of the operation $\oplus$ we get:

$$(U + V) + W = [[U \oplus V] \oplus W] = [(U \oplus V) \oplus W]$$
$$= [U \oplus (V \oplus W)] = [U \oplus [V \oplus W]] = U + (V + W).$$

The associativity for $\vee$ and the distributivity law can be proven in a similar way using (8) as well as (9). $\square$

Note that (6) can be rephrased, with respect to E_{ciw}, as the inequation $0 \leqslant x$, and (7) as $x \geqslant 0$. Also, writing $\vee$ for $\wedge$, Eq. (7) takes the form $x \vee 0 = 0$ that one should have if $\vee$ is considered to be the signature symbol instead of $\wedge$.

For any structure $\mathbf{A}$, we use $\mathscr{V}(\mathbf{A})$ to denote the variety generated by $\mathbf{A}$, i.e., the class of algebras that satisfy the equations that hold in $\mathbf{A}$. Our order of business will now be to offer concrete descriptions of the finitely generated free algebras in the varieties generated by $\mathbf{Z}_\vee$, $\mathbf{N}_\vee$ and $\mathbf{N}_\wedge$.

Theorem 23. *For each $n \geqslant 0$, $C(\mathbb{N}^n)$ is freely generated in $\mathscr{V}(\mathbf{Z}_\vee)$ by the sets $[\bar{p}_i]$, $i \in [n]$.*

Proof. Each $C \in C(\mathbb{N}^n)$ may be written as $\bigvee_{\bar{c} \in C} \{\bar{c}\}$, and, for each $\bar{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$, it holds that $\{\bar{c}\} = \sum_{i=1}^n c_i[\bar{p}_i]$. It follows that $C(\mathbb{N}^n)$ is generated by the sets $[\bar{p}_i]$. Suppose now that h is a function $\{[\bar{p}_1], \ldots, [\bar{p}_n]\} \to \mathbb{Z}$, say $h : [\bar{p}_i] \mapsto x_i$, $i \in [n]$. We need to show that h uniquely extends to a homomorphism $h^\sharp : C(\mathbb{N}^n) \to \mathbf{Z}_\vee$. For each set $C \in C(\mathbb{N}^n)$, define

$$h^\sharp(C) = \bigvee_{\bar{c} \in C} \bar{c} \cdot \bar{x},$$

where $\bar{x}$ is the vector $(x_1, \ldots, x_n)$. As an immediate consequence of the definition of $h^\sharp$, we have that $h^\sharp([\bar{p}_i]) = x_i$, for all $i \in [n]$, and that $h^\sharp(0) = 0$. Also, if F is a non-empty finite subset of $\mathbb{N}^n$, then, by Proposition 17,

$$h^\sharp([F]) = \bigvee_{\bar{u} \in [F]} \bar{u} \cdot \bar{x}$$
$$= \bigvee_{\bar{u} \in F} \bar{u} \cdot \bar{x}.$$

Thus, for $C, D \in C(\mathbb{N}^n)$,

$$h^\sharp(C \vee D) = h^\sharp([C \cup D])$$
$$= \bigvee_{\bar{u} \in C \cup D} \bar{u} \cdot \bar{x}$$

$$= \bigvee_{\bar{c} \in C} \bar{c} \cdot \bar{x} \vee \bigvee_{\bar{d} \in D} \bar{d} \cdot \bar{x}$$
$$= h^{\sharp}(C) \vee h^{\sharp}(D).$$

Also,

$$h^{\sharp}(C + D) = h^{\sharp}([C \oplus D])$$
$$= \bigvee_{\bar{u} \in C \oplus D} \bar{u} \cdot \bar{x}$$
$$= \bigvee_{\bar{c} \in C} \bar{c} \cdot \bar{x} + \bigvee_{\bar{d} \in D} \bar{d} \cdot \bar{x}$$
$$= h^{\sharp}(C) + h^{\sharp}(D),$$

since $+$ distributes over $\vee$. This proves that $h^{\sharp}$ is a homomorphism. Moreover, $h^{\sharp}$ is the only extension of h since $C(\mathbb{N}^n)$ is generated by the sets $[\bar{p}_i]$, $i \in [n]$.

We still need to verify that $C(\mathbb{N}^n)$ belongs to $\mathscr{V}(\mathbf{Z}_\vee)$. To this end, for each $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, let $h_{\bar{x}}$ denote the homomorphism $C(\mathbb{N}^n) \to \mathbf{Z}_\vee$ described above taking $[\bar{p}_i]$ to x_i, $i \in [n]$. If $C, D \in C(\mathbb{N}^n)$ with $\bar{d} \in D - C$, say, then, by Proposition 17,

$$\bar{d} \cdot \bar{y} > \bigvee_{\bar{c} \in C} \bar{c} \cdot \bar{y},$$

for some $\bar{y} \in \mathbb{Z}^n$. Thus, it holds that $h_{\bar{y}}(D) > h_{\bar{y}}(C)$. It follows that the target tupling of the functions $h_{\bar{x}}$ is an injective homomorphism from $C(\mathbb{N}^n)$ to a direct power of $\mathbf{Z}_\vee$, proving that $C(\mathbb{N}^n)$ belongs to $\mathscr{V}(\mathbf{Z}_\vee)$. $\square$

Free algebras in $\mathscr{V}(\mathbf{N}_\vee)$ and $\mathscr{V}(\mathbf{N}_\wedge)$ have a similar description.

Theorem 24. *For each $n \geqslant 0$, $CI(\mathbb{N}^n)$ is freely generated in $\mathscr{V}(\mathbf{N}_\vee)$ by the sets $(\bar{p}_i]$, $i \in [n]$. Moreover, $CF(\mathbb{N}^n)$ is freely generated in $\mathscr{V}(\mathbf{N}_\wedge)$ by the sets $[\bar{p}_i)$, $i \in [n]$.*

Proof. The argument is similar to the proof of Theorem 23. We outline the proof of the second claim.

First, the convex filters $[\bar{p}_i)$, $i \in [n]$, form a generating system of $CF(\mathbb{N}^n)$. This follows from Remark 16 by noting that if U is the convex filter generated by the finite non-empty set F, then

$$U = \bigwedge_{\bar{c} \in F} [\bar{c}).$$

Moreover, for each $\bar{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$, it holds that

$$[\bar{c}) = \sum_{i=1}^n c_i [\bar{p}_i).$$

Let h be the assignment $[\bar{p}_i) \mapsto x_i \in \mathbb{N}$, $i \in [n]$. By the above argument, h has at most one extension to a homomorphism $h^{\sharp} : CF(\mathbb{N}^n) \to \mathbf{N}_\wedge$. Define $h^{\sharp}$ by

$$h^{\sharp}(U) = \bigwedge_{\bar{c} \in U} \bar{c} \cdot \bar{x},$$

where $\bar{x} = (x_1, \ldots, x_n)$. Thus, $h^{\sharp}(U)$ is the least element of the set

$$\{\bar{c} \cdot \bar{x}: \ \bar{c} \in U\} \subseteq \mathbb{N}.$$

(Actually, the above definition makes sense for any non-empty set $U \subseteq \mathbb{N}^n$.) The proof of the fact that $h^{\sharp}$ preserves the operations follows the lines of the similar verification in the proof of Theorem 23. One uses Proposition 19 in lieu of Proposition 17. The fact that $CF(\mathbb{N}^n)$ belongs to $\mathscr{V}(\mathbf{N}_\wedge)$ can be shown as the corresponding fact in the proof of Theorem 23. $\square$

Remark 25. A complete proof of the first part of Theorem 24 is given in [1].

Remark 26. In any variety, any infinitely generated free algebra is the direct limit of finitely generated free algebras. More specifically, for any cardinal number κ, the free algebra on κ generators in $\mathscr{V}(\mathbf{Z}_\vee)$ can be described as an algebra of finite non-empty convex sets in $\mathbb{N}^\kappa$ consisting of vectors whose components, with a finite number of exceptions, are all zero. The free algebras in $\mathscr{V}(\mathbf{N}_\vee)$ and $\mathscr{V}(\mathbf{N}_\wedge)$ have similar descriptions using finitely generated convex ideals and filters, respectively.

Since $\mathbf{N}_\vee$ is a subalgebra of $\mathbf{Z}_\vee$, we have that $\mathscr{V}(\mathbf{N}_\vee) \subseteq \mathscr{V}(\mathbf{Z}_\vee)$. Also, since $\mathbf{N}_\wedge$ is isomorphic to the subalgebra $\mathbf{N}_\vee^-$ of $\mathbf{Z}_\vee$, it holds that $\mathscr{V}(\mathbf{N}_\wedge) \subseteq \mathscr{V}(\mathbf{Z}_\vee)$.

Definition 27. Let $\mathscr{V}$ and $\mathscr{V}'$ be two varieties of algebras such that the signature of $\mathscr{V}$ extends that of $\mathscr{V}'$. Let E be a collection of equations in the language of $\mathscr{V}$. We say that $\mathscr{V}$ is *axiomatized over* $\mathscr{V}'$ *by* E if the collection of equations that hold in $\mathscr{V}'$ together with E form a basis for the identities of $\mathscr{V}$. We say that $\mathscr{V}$ has a *finite axiomatization relative to* $\mathscr{V}'$ if $\mathscr{V}$ is axiomatized over $\mathscr{V}'$ by some finite set of equations E.

In the next result we show that both $\mathscr{V}(\mathbf{N}_\vee)$ and $\mathscr{V}(\mathbf{N}_\wedge)$ possess a finite axiomatization relative to $\mathscr{V}(\mathbf{Z}_\vee)$. Of course, $\mathscr{V}(\mathbf{Z}_\vee)$ is just $\mathscr{V}(\mathbf{Z}_\wedge)$, since $\mathbf{Z}_\vee$ and $\mathbf{Z}_\wedge$ are isomorphic.

Theorem 28. *The following statements hold*:
(1) $\mathscr{V}(\mathbf{N}_\vee)$ *is axiomatized over* $\mathscr{V}(\mathbf{Z}_\vee)$ *by the equation* $x \vee 0 = x$.
(2) $\mathscr{V}(\mathbf{N}_\wedge)$ *is axiomatized over* $\mathscr{V}(\mathbf{Z}_\wedge)$ *by the equation* $x \wedge 0 = 0$.

Proof. Suppose that n is any non-negative integer. By Theorem 23, the map $[\bar{p}_i] \mapsto (\bar{p}_i)$, $i \in [n]$, extends to a unique homomorphism $h: C(\mathbb{N}^n) \to CI(\mathbb{N}^n)$. Comparing the definitions of the operations in $C(\mathbb{N}^n)$ and $CI(\mathbb{N}^n)$ and using the first part of Proposition 13, it is easy to see that h is in fact the function $U \mapsto [(U)]$, $U \in C(\mathbb{N}^n)$. We prove that the kernel of h is the least congruence on $C(\mathbb{N}^n)$ such that the quotient satisfies (6), which implies the first claim.

As $\bar{0}$ is contained in $[(U)]$ for every non-empty set U of vectors in $\mathbb{N}^n$, the kernel of h satisfies (6). To complete the proof, we need to show that if φ is a homomorphism

$C(\mathbb{N}^n) \to \mathbf{B}$, where $\mathbf{B}$ is in $\mathscr{V}_{\mathrm{ciw}}^+$, then the kernel of h is included in the kernel of φ. But, since $\mathbf{B}$ is in $\mathscr{V}_{\mathrm{ciw}}^+$, for every $U \in C(\mathbb{N}^n)$, it holds that

$$\varphi(U) = \bigvee_{\bar{u} \in U} \varphi([\bar{u}]) = \bigvee_{\bar{u} \in (U]} \varphi([\bar{u}]) = \bigvee_{\bar{u} \in [(U]]} \varphi([\bar{u}]) = \varphi([(U]]),$$

so that if $[(U]] = [(V]]$, then $\varphi(U) = \varphi(V)$. (Here, the second equality follows since $\mathbf{B}$ is a positive ciw-semiring and thus satisfies (1), while the third equality follows from Proposition 17.)

The proof of the second claim follows similar lines, using the second part of Proposition 13. $\quad\square$

3.4. Non-finite axiomatizability results

Our order of business in this section is to show that the varieties generated by the ciw-semirings $\mathbf{Z}_\vee$ and $\mathbf{N}_\wedge$ are not finitely based. Our starting points are the results in [1] to the effect that the variety $\mathscr{V}(\mathbf{N}_\vee)$ is not finitely based. These we restate below for ease of reference and for completeness.

Theorem 29. *The following statements hold:*
(1) The variety $\mathscr{V}(\mathbf{N}_\vee)$ is not finitely based.
(2) For every $n \in \mathbb{N}$, the collection of all the inequations in at most n variables that hold in $\mathscr{V}(\mathbf{N}_\vee)$ does not form an equational basis for it.

We begin by using these results, and the first part of Theorem 28 to prove:

Theorem 30. *The variety $\mathscr{V}(\mathbf{Z}_\vee)$ is not finitely based. Moreover, $\mathscr{V}(\mathbf{Z}_\vee)$ has no axiomatization by equations in a bounded number of variables, i.e., there exists no natural number n such that the collection of all equations in at most n variables that hold in $\mathscr{V}(\mathbf{Z}_\vee)$ forms an equational basis for $\mathscr{V}(\mathbf{Z}_\vee)$.*

Proof. The first claim is an immediate consequence of the second. To prove the second claim, we argue as follows. Assume, towards a contradiction, that there is a non-zero natural number n such that the collection of all equations in at most n variables that hold in $\mathscr{V}(\mathbf{Z}_\vee)$ forms an equational basis for $\mathscr{V}(\mathbf{Z}_\vee)$. Then, by the first part of Theorem 28, this collection of equations together with (6) forms an equational basis for $\mathscr{V}(\mathbf{N}_\vee)$ consisting of equations in at most n variables. However, this contradicts the second statement in Theorem 29. $\quad\square$

An alternative proof of the above result will be sketched in Remark 46 to follow.

We now proceed to apply the results that we have developed so far to the study of the axiomatizability of the equational theory of the algebra $\mathbf{N}_\wedge$.

Our aim in the remainder of this section is to prove the following result to the effect that the variety $\mathscr{V}(\mathbf{N}_\wedge)$ has no finite equational basis.

Theorem 31. *The variety $\mathscr{V}(\mathbf{N}_\wedge)$ has no finite (equational) axiomatization, i.e., there is no finite set E of equations, which hold in $\mathscr{V}(\mathbf{N}_\wedge)$, and such that for all*

terms t_1, t_2,

$$\mathscr{V}(\mathbf{N}_\wedge) \models t_1 = t_2 \ \textit{iff} \ E \ \textit{proves} \ t_1 = t_2.$$

To prove Theorem 31, we begin by noting that the following equations $e_n^\wedge$ hold in $\mathbf{N}_\wedge$, for each $n \geqslant 2$:

$$e_n^\wedge: \ r_n \wedge s_n = s_n, \tag{10}$$

where

$$r_n = x_1 + \cdots + x_n$$
$$s_n = (2x_1 + x_3 + x_4 + \cdots + x_{n-1} + x_n)$$
$$\wedge (x_1 + 2x_2 + x_4 + \cdots + x_{n-1} + x_n)$$
$$\vdots$$
$$\wedge (x_1 + x_2 + x_3 + \cdots + x_{n-2} + 2x_{n-1})$$
$$\wedge (x_2 + x_3 + x_4 + \cdots + x_{n-1} + 2x_n).$$

In what follows, we shall define a sequence of ciw-semirings $\mathbf{B}_n$ ($n \geqslant 3$) such that the following holds:

For any finite set E of equations which hold in $\mathscr{V}(\mathbf{N}_\wedge)$, there is an $n \geqslant 3$ such that

$$\mathbf{B}_n \models E \ \text{but} \ \mathbf{B}_n \not\models e_n^\wedge.$$

In fact, as we shall see in due course, the algebra construction that we now proceed to present also yields the following stronger result.

Theorem 32. *There exists no natural number n such that the collection of all equations in at most n variables that hold in $\mathscr{V}(\mathbf{N}_\wedge)$ forms an equational basis for $\mathscr{V}(\mathbf{N}_\wedge)$.*

The remainder of this section will be devoted to a proof of Theorem 32. We begin with some preliminary definitions and results that will pave the way to the construction of the algebras $\mathbf{B}_n$.

Definition 33. The *weight* of a vector $\bar{v} = (v_1, \ldots, v_n)$ in $\mathbb{N}^n$ is defined as $v_1 + \cdots + v_n$. The *weight* of a non-empty set $U \subseteq \mathbb{N}^n$ is the minimum of the weights of the vectors in U.

The subsequent lemma collects those properties of the notion of weight defined above that will be needed in the technical developments to follow.

Lemma 34. The following statements hold:
(1) *Let $\bar{u}$ and $\bar{v}$ be vectors in $\mathbb{N}^n$. Then the weight of $\bar{u} + \bar{v}$ is the sum of the weights of $\bar{u}$ and $\bar{v}$.*

(2) *Let $\bar{u}$ be a vector in $\mathbb{N}^n$.*
   (a) *If $\bar{u} < \bar{v}$ in $\mathbb{N}^n$, with respect to the pointwise order, then the weight of $\bar{u}$ is less than the weight of $\bar{v}$.*
   (b) *Assume that $\bar{u}$ is a convex linear combination $\lambda_1 \bar{v}_1 + \cdots + \lambda_k \bar{v}_k$ of vectors $\bar{v}_1, \ldots, \bar{v}_k \in \mathbb{N}^n$. Let w_i denote the weight of vector $\bar{v}_i$, for every $i \in [k]$. Then the weight of $\bar{u}$ is $\lambda_1 w_1 + \cdots + \lambda_k w_k$. Moreover, the weight of $\bar{u}$ is greater than, or equal to, the minimum of the w_i and less than, or equal to, the maximum of the w_i.*
(3) *Let U be a non-empty subset of $\mathbb{N}^n$. Then the following statements hold.*
   (a) *The weight of U is equal to that of $[U]$, $[U)$ and $[[U)]$.*
   (b) *Let k be the weight of U. Then, a weight k vector belongs to $[[U)]$ iff it is a convex linear combination of some vectors of weight k in U.*
(4) *Let U, V be a non-empty subset of $\mathbb{N}^n$. Then the weight of $U \oplus V$ is equal to the sum of the weights of U and V.*

Proof. We only give the proofs of statements 3 and 4.

- **Proof of Statement 3.** Let U be a non-empty subset of $\mathbb{N}^n$. We consider Statements 3a and 3b in turn.
- **Proof of Statement 3a.** Since U is included in both $[U]$ and $[U)$, it follows that the weights of both $[U]$ and $[U)$ are no larger than that of U. As for every vector $\bar{v}$ contained in $[U)$ there is a vector $\bar{u}$ in U such that $\bar{u} \leqslant \bar{v}$ with respect to the pointwise order in $\mathbb{N}^n$, by Statement 2a of the lemma we have that the weight of U is less than, or equal to, the weight of $[U)$. Thus $[U)$ and U have equal weight.

  We now argue that $[U]$ and U also have equal weight. To this end, recall that, by Proposition 12, every vector $\bar{v} \in [U]$ is a convex linear combination of vectors $\bar{u}_1, \ldots, \bar{u}_k \in U$. By Statement 2b of the lemma, the weight of $\bar{v}$ is larger than, or equal to, the minimum weight of the $\bar{u}_i$ ($i \in [k]$). Thus, for every $\bar{v} \in [U]$ there is a vector $\bar{u} \in U$ whose weight is no larger than that of $\bar{v}$. It follows that $[U]$ and U have equal weight, as claimed.

  As an immediate consequence of the claims proven above, we have that $[[U)]$ and U also have equal weight.

- **Proof of Statement 3b.** We first establish the "only if" implication. To this end, assume that k is the weight of U, and that $\bar{u}$ is a vector of weight k in $[[U)]$. As $\bar{u}$ is contained in $[[U)]$, Proposition 12 entails that, with respect to the pointwise order, $\bar{u}$ is above a convex linear combination of vectors $\bar{u}_1, \ldots, \bar{u}_l \in U$. As U has weight k, the weight of each of the $\bar{u}_i$ ($i \in [l]$) is larger than, or equal to, k. Moreover, as k is larger than, or equal to, a convex linear combination of the weights of the $\bar{u}_i$ ($i \in [l]$), it follows that each of the $\bar{u}_i$ ($i \in [l]$) has weight exactly k. Thus, by statements 2a and 2b of the lemma, $\bar{u}$ is a convex linear combination of the vectors $\bar{u}_1, \ldots, \bar{u}_l$, which was to be shown.

  Conversely, if $\bar{u}$ is a convex linear combination of vectors of weight k in U, then $\bar{u}$ has weight k (Statement 2b of the lemma), and is contained in $[[U)]$.

- **Proof of Statement 4.** Let U, V be a non-empty subset of $\mathbb{N}^n$. Then any vector of minimum weight in $U \oplus V$ is the sum of two vectors of minimum weight in U and V. $\square$

We introduce the following notations for some vectors in $\mathbb{N}^n$ related to the equation $e_n^\wedge$:

$$\bar{\delta} = (1, \ldots, 1)$$
$$\bar{\gamma}_1 = (2, 0, 1, 1, \ldots, 1, 1)$$
$$\bar{\gamma}_2 = (1, 2, 0, 1, \ldots, 1, 1)$$
$$\vdots$$
$$\bar{\gamma}_{n-1} = (1, 1, 1, 1, \ldots, 2, 0)$$
$$\bar{\gamma}_n = (0, 1, 1, 1, \ldots, 1, 2),$$

so that in $\bar{\gamma}_i$ ($i \in [n]$), the 2 is on the ith position and is followed by a 0. (Of course, we assume that the first position follows the nth.) All other components are 1. Note that

$$\bar{\delta} = \frac{1}{n} \bar{\gamma}_1 + \cdots + \frac{1}{n} \bar{\gamma}_n. \tag{11}$$

Thus, $\bar{\delta}$ belongs to the convex filter generated by the vectors $\bar{\gamma}_i$ ($i \in [n]$). Moreover, the system consisting of any n of the vectors $\bar{\delta}, \bar{\gamma}_1, \ldots, \bar{\gamma}_n$ is linearly independent (cf. [1, Lemma 5.2]).

We define:

$$\Gamma = [[\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\})]],$$
$$\Delta = \Gamma - \{\bar{\delta}\},$$

so that Γ is the convex filter generated by the $\bar{\gamma}_i$ ($i \in [n]$). By (11), the set Δ is not a convex filter. It follows from the following lemma that the only vectors of weight n in Γ are $\bar{\delta}$ and the $\bar{\gamma}_i$ ($i \in [n]$).

Lemma 35. *Suppose that a non-empty subset U of $\mathbb{N}^n$ satisfies*:
(1) *The weight of U is greater than, or equal to, n.*
(2) *Any vector of weight n in U belongs to the set $\{\bar{\delta}, \bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$.*
Then every vector of weight n in $[[U]]$ lies in the set $\{\bar{\delta}, \bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$. Moreover, $\bar{\gamma}_i \in [[U]]$ iff $\bar{\gamma}_i \in U$, and $\bar{\delta} \in [[U]]$ iff $\bar{\delta} \in U$ or $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\} \subseteq U$.

Proof. Suppose that $\bar{u} \in [[U]]$ has weight n. Then, by Lemma 34, $\bar{u}$ is a convex linear combination of weight n vectors in U, and hence a convex linear combination of the vectors $\bar{\delta}, \bar{\gamma}_1, \ldots, \bar{\gamma}_n$. It follows that no component of $\bar{u}$ is greater than 2 and at most one component is 0. Moreover, if the ith component of $\bar{u}$ is 2, for some i, then necessarily $\bar{u} = \bar{\gamma}_i$. Suppose that $\bar{u} = \bar{\delta}$. Then, since any n of the vectors $\bar{\delta}, \bar{\gamma}_1, \ldots, \bar{\gamma}_n$ form a linearly

independent system, and since $\bar{\delta}$ is a convex linear combination of the $\bar{\gamma}_i$ ($i \in [n]$), it follows that either $\bar{\delta} \in U$ or $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\} \subseteq U$. $\square$

Corollary 36. *The convex filter Γ has $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$ as its unique minimal generating set.*

Proof. Since $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$ generates Γ, it is sufficient to argue that the set $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$ must be included in every set G that generates Γ as a convex filter. To this end, assume that $[[G]] = \Gamma$. Then, by Lemma 34(3a), G has weight n. Now each $\bar{\gamma}_i$ is in $[[G]]$ and is therefore, by Lemma 34(3b), a convex linear combination of some vectors of weight n in G. But by Lemma 35, any such vector is $\bar{\delta}$ or some of the $\bar{\gamma}_i$. Moreover, again by Lemma 35, each $\bar{\gamma}_i$ is in G, which was to be shown. $\square$

Lemma 37. *Suppose that F is a convex filter in $\mathbb{N}^n$ properly included in Γ. Then $F - \{\bar{\delta}\}$ is also a convex filter.*

Proof. By our assumptions, the set $F - \{\bar{\delta}\}$ is a convex filter if $\bar{\delta} \notin F$, so assume $\bar{\delta} \in F$. Now observe that $F - \{\bar{\delta}\}$ is a convex filter unless $\bar{\delta} \in [[F - \{\bar{\delta}\}]]$. But since each vector of weight n in $F - \{\bar{\delta}\}$ is one of the $\bar{\gamma}_i$, and since at least one $\bar{\gamma}_i$ is not in $F - \{\bar{\delta}\}$, by Lemma 35 we have that $\bar{\delta} \notin [[F - \{\bar{\delta}\}]]$. $\square$

We now proceed to define the algebras $\mathbf{B}_n = (B_n, \wedge, +, 0)$, for every $n > 1$.

Let B_n consist of the non-empty convex filters in $\mathbb{N}^n$ and the set $\Delta = \Gamma - \{\bar{\delta}\}$. The following results will allow us to endow B_n with the structure of a ciw-semiring.

Corollary 38. *If F is a convex filter, then $F \cap \Delta \in B_n$.*

Proof. Since Δ is included in Γ, it holds that $F \cap \Delta = (F \cap \Gamma) \cap \Delta$. Thus we may assume that F is included in Γ. If $F = \Gamma$, the intersection is Δ. Otherwise F is properly included in Γ and $F \cap \Delta = F - \{\bar{\delta}\} \in B_n$, by the previous lemma. $\square$

Proposition 39. *If the intersection of a family of sets in B_n is not empty, then the intersection is in B_n.*

Proof. Let U_i, $i \in I$, be a family of sets in B_n such that $U = \bigcap_{i \in I} U_i$ is non-empty. If each U_i is different from Δ, then U is a non-empty convex filter and is thus in B_n. Otherwise, we have that

$$U = \Delta \cap \bigcap\{U_i: \ i \in I, \ U_i \neq \Delta\},$$

and the result follows by the previous corollary. $\square$

For each non-empty set $U \subseteq \mathbb{N}^n$, let $\mathbf{cl}(U)$ denote the least set in B_n containing U. (This set exists in light of the proposition above.) For each $U, V \in B_n$,

we define

$$U + V = \mathbf{cl}(U \oplus V),$$
$$U \wedge V = \mathbf{cl}(U \cup V).$$

Moreover, we define the constant 0 to be the set $[[\bar{0}]] = [\bar{0}) = \mathbb{N}^n$. This completes the definition of the algebra $\mathbf{B}_n = (B_n, \wedge, +, 0)$.

Recall that $CF(\mathbb{N}^n)$ is freely generated by the sets $[\bar{p}_i)$, $i \in [n]$, in the variety generated by $\mathbf{N}_\wedge$. The following result will be useful in the proof of Proposition 45 to follow.

Lemma 40. *The function given by $\varDelta \mapsto \Gamma$ and $U \mapsto U$, if $U \neq \varDelta$, defines a homomorphism $\mathbf{B}_n \to CF(\mathbb{N}^n)$.*

Proof. To prove the lemma we have to show that the assignment that defines the mapping given above preserves the operations. Below we only give the proof for the operation $+$ as the proof for $\vee$ is similar. Towards proving the statement for the $+$ operation, assume that $U, V \in \mathbf{B}_n$ and that Z is their sum in $\mathbf{B}_n$. We now proceed by case analysis on the form of U and V.

Case $U, V \neq \varDelta$: Then either Z is the sum $U + V$ taken in $CF(\mathbb{N}^n)$ or $Z = \varDelta$ in which case the sum $U + V$ taken in $CF(\mathbb{N}^n)$ is Γ, since Γ is the least convex filter containing $\varDelta$. Thus, in this case, the assignment preserves $+$.

Case $U = \varDelta, V \neq \varDelta$: Then $Z = \varDelta$ if $V = 0$, and Z is the sum $\Gamma + V$ taken in $CF(\mathbb{N}^n)$ otherwise. In either case the assignment preserves $+$.

Case $V = \varDelta, U \neq \varDelta$: Symmetric to the previous case.

Case $U = V = \varDelta$: Then Z is the least convex filter containing $\varDelta \oplus \varDelta$, and is thus equal to the sum $\Gamma + \Gamma$ taken in $CF(\mathbb{N}^n)$, which was to be shown. $\square$

To show that $\mathbf{B}_n$ is a ciw-semiring, we need:

Lemma 41. *When $n \geqslant 3$, the filter Γ has no decomposition in $CF(\mathbb{N}^n)$ into the sum of two non-zero convex filters. Similarly, when $n \geqslant 3$, neither Γ nor $\varDelta$ has a non-trivial decomposition in $\mathbf{B}_n$ into the sum of two non-zero sets.*

Proof. First we work in $CF(\mathbb{N}^n)$.

Assume, towards a contradiction, that $n \geqslant 3$, F and G are non-empty convex filters with $F + G = \Gamma$ in $CF(\mathbb{N}^n)$, but $F, G \neq \mathbb{N}^n$. Let k denote the weight of F and ℓ the weight of G. Then $k, \ell > 0$ and $k + \ell = n$ (Statements 3a and 4 in Lemma 34). Let F' denote the set of all vectors of weight k in F, and define $G' \subseteq G$ in similar fashion. Then, with respect to the pointwise order, every vector in $F + G = \Gamma$ is greater than, or equal to, a convex linear combination of vectors in $F \oplus G$. But, since every vector in $F \oplus G$ has weight at least n, it follows that a weight n vector is in $F + G$ if and only if it is a convex linear combination of weight n vectors in $F \oplus G$, i.e. of vectors in $F' \oplus G'$. Thus, each vector in the set $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n, \bar{\delta}\}$ is a convex linear combination of vectors in $F' \oplus G'$, and since $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n, \bar{\delta}\}$ is a set of generators for Γ, it follows

that $F' \oplus G'$ is also a set of generators for Γ. Thus, by Corollary 36, we have that

$$\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\} \subseteq F' \oplus G'.$$

Moreover, it holds that

$$F' \oplus G' \subseteq \{\bar{\gamma}_1, \ldots, \bar{\gamma}_n, \bar{\delta}\}. \tag{12}$$

This follows because $F' \oplus G'$ is a set of vectors of weight n included in Γ, and the only vectors of weight n in Γ are $\bar{\delta}$ and the $\bar{\gamma}_i$ ($i \in [n]$).

Suppose that F', say, contains a vector $\bar{u}$ which has a component equal to 2 on its ith position. Then G' contains a single vector $\bar{v}$. Moreover, the vector $\bar{v}$ has 0 on its ith and $(i+1)$st position, and on those positions where $\bar{u}$ contains a 1, and 1 in all other positions. Since $\bar{v} \neq 0$, for some j its jth component is not 0. But then $\bar{\gamma}_{j-1}$ is not contained in $F' \oplus G'$, contradicting (12). Thus, F' contains no vector having a component equal to 2, and similarly for G'.

Since the complex sum of F' and G' contains the vectors $\bar{\gamma}_1$ and $\bar{\gamma}_2$, there are vectors $\bar{w}_1, \bar{w}_2 \in F'$ and $\bar{v}_1, \bar{v}_2 \in G'$ such that

$$\bar{w}_1 + \bar{v}_1 = \bar{\gamma}_1 \quad \text{and} \quad \bar{w}_2 + \bar{v}_2 = \bar{\gamma}_2.$$

This means that, for some $b_3, \ldots, b_n \in \{0, 1\}$,

$$\bar{w}_1 = (1, 0, b_3, \ldots, b_n)$$

and

$$\bar{v}_1 = (1, 0, \tilde{b}_3, \ldots, \tilde{b}_n).$$

where $\tilde{b}$ denotes the complement of b, for every $b \in \{0, 1\}$. Similarly, since $n \geqslant 3$, there are $c_1, c_4, \ldots, c_n \in \{0, 1\}$ such that

$$\bar{w}_2 = (c_1, 1, 0, c_4, \ldots, c_n)$$

and

$$\bar{v}_2 = (\tilde{c}_1, 1, 0, \tilde{c}_4, \ldots, \tilde{c}_n).$$

It is now easy to see that if $\bar{w}_1 + \bar{v}_2$ is in $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n, \bar{\delta}\}$, then $\bar{w}_2 + \bar{v}_1$ is not. Indeed, if $\bar{w}_1 + \bar{v}_2$ is in $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n, \bar{\delta}\}$, then $\tilde{c}_1 = 0$, so that $c_1 = 1$. Thus the first two components of $\bar{w}_2 + \bar{v}_1$ are 2 and 1, respectively, contradicting (12).

Assume now that $F + G \in \{\Gamma, \Delta\}$ in $\mathbf{B}_n$. If $F, G \in CF(\mathbb{N}^n)$, then $F + G = \Gamma$ in $CF(\mathbb{N}^n)$. By the first claim, this is possible only if F or G is 0. If $F = \Delta$, say, then G must be 0, or else $F \oplus G$ would only contain vectors whose weight is greater than n. This completes the proof. $\square$

Remark 42. When $n = 2$, the set Δ does not have a non-trivial representation as the sum of two non-zero elements of $\mathbf{B}_n$, but we have

$$[[\{(1, 0), (0, 1)\})]] + [[\{(1, 0), (0, 1)\})]] = \Gamma$$

both in $\mathbf{B}_n$ and in $CF(\mathbb{N}^n)$.

Proposition 43. *If $n \geqslant 3$, then $\mathbf{B}_n$ is a ciw-semiring satisfying $x \wedge 0 = 0$.*

Proof. It is obvious that both binary operations are commutative and that $\wedge$ is idempotent. Also, the equation $x \wedge 0 = 0$ holds. The fact that $\wedge$ is associative follows from general properties of closure operators. In fact,

$$(A \wedge B) \wedge C = \mathbf{cl}(A \cup B \cup C) = A \wedge (B \wedge C),$$

for all $A, B, C \in B_n$. The facts that also

$$(A + B) + C = A + (B + C),$$
$$(A \wedge B) + C = (A + C) \wedge (B + C)$$

hold follow by Lemma 41. The only way that these equations can fail is that one side is Δ and the other is Γ. But in that case one of A, B, C is 0, by Lemma 41 and since in $CF(\mathbb{N}^n)$, and in $\mathbf{B}_n$, if $A \wedge B = 0$ then $A = 0$ or $B = 0$, and then both equations obviously hold. $\quad\square$

Remark 44. For all $A, B, C \in B_n$, we have that

$$A + B + C = \mathbf{cl}(A \oplus B \oplus C). \tag{13}$$

Indeed, if $A + B + C$ is different from Δ, then it is a convex filter and, as in the proof of Proposition 22, we can argue that it equals $[[A \oplus B \oplus C]] = \mathbf{cl}(A \oplus B \oplus C)$. If $A + B + C$ equals Δ, then, by Lemma 41 and Remark 42, exactly one of A, B and C is Δ and the other two are 0. In this case, we have that

$$\mathbf{cl}(A \oplus B \oplus C) = \mathbf{cl}(\Delta) = \Delta,$$

which was to be shown.

Equality (13) will be used implicitly in the proof of the following result, which is the crux of the proof of Theorem 32.

Proposition 45. *For each $n \geqslant 3$, the algebra $\mathbf{B}_n$ satisfies any equation in at most $n-1$ variables which holds in $\mathbf{N}_\wedge$.*

Proof. It suffices to show that $\mathbf{B}_n \models t \geqslant t'$ for any simple $\wedge$-inequation $t \geqslant t'$ such that $\mathbf{N}_\wedge \models t \geqslant t'$ and both t and t' contain the same at most $m < n$ variables, so that $t = t(x_1, \ldots, x_m)$ and $t' = t'(x_1, \ldots, x_m)$, say. Indeed, since by Proposition 43, $\mathbf{B}_n$ is a ciw-semiring satisfying $x \wedge 0 = 0$, any variable occurring in t' must occur in t. Moreover, we can set each variable occurring in t, which does not occur in t', to 0. (This matter will be discussed in more detail in Section 4.1.) By Lemma 40, we only need to show that for all $U_1, \ldots, U_m$ in B_n, it is not possible that $\bar{\delta} \in V$ and $\Delta = V'$, where $V = t(U_1, \ldots, U_m)$ and $V' = t'(U_1, \ldots, U_m)$.

Assume, towards a contradiction, that for some $U_i \in B_n$, $i \in [m]$, we have $\bar{\delta} \in V$ and $\Delta = V'$, and that $t \geqslant t'$ is a simple inequation in fewest variables for which this holds.

Note that this implies that $U_i \neq 0$ for every $i \in [m]$. Moreover, the weight of each U_i ($i \in [m]$) is at most n. In fact, since $\bar{\delta} \in V = t(U_1, \ldots, U_m)$ in $\mathbf{B}_n$, we have that V is a convex filter whose weight is at most n. As V is a convex filter, by Lemma 40, in $CF(\mathbb{N}^n)$ it holds that

$$V = t(U_1', \ldots, U_m'),$$

where, for every $i \in [m]$,

$$U_i' = \begin{cases} \Gamma & \text{if } U_i = \Delta, \\ U_i & \text{otherwise.} \end{cases}$$

(Note that, for every $i \in [m]$, the weight of U_i is equal to that of U_i'.) Since $t(x_1, \ldots, x_m)$ is a "linear combination" of the variables $x_1, \ldots, x_m$, it follows by Lemma 34(3–4) that the weight of each U_i', and thus of each U_i, is at most n, as claimed. Also, if the weight of some U_i is n, then $m = 1$ and $t = x_1$. Since $t \geqslant t'$ holds in $\mathbf{N}_\wedge$ and t' contains exactly x_1, it follows that $t' = x_1$ or $t' = 0$ modulo the equations of ciw-semirings and equation $x \wedge 0 = 0$. But then $t \geqslant t'$ holds in $\mathbf{B}_n$, contrary to our assumption. Thus, each U_i is different from Δ and is thus a non-empty convex filter. Since $V \neq \Delta$, it holds that $V = t(U_1, \ldots, U_m)$ also in $CF(\mathbb{N}^n)$. Moreover, we have that $t'(U_1, \ldots, U_m) = \Gamma$ in $CF(\mathbb{N}^n)$. By Lemma 41, it holds that $V \neq \Gamma$. Since $t \geqslant t'$ holds in $CF(\mathbb{N}^n)$, we have that $V \subseteq \Gamma$. Thus V is a proper subset of Γ.

Now write $t' = t_1' \wedge \cdots \wedge t_k'$, where the t_i' are linear combinations of the variables $x_1, \ldots, x_m$. For each $i \in [k]$, let $V_i' = t_i'(U_1, \ldots, U_m)$ in $\mathbf{B}_n$. Each V_i' is included in Δ, and is in fact a proper subset of Δ, since Δ has no non-trivial decomposition in $\mathbf{B}_n$ into the sum of two sets (Lemma 41). Thus, each V_i' is a non-empty convex filter, and $V_i' = t_i'(U_1, \ldots, U_m)$ also in $CF(\mathbb{N}^n)$. Call a t_i', and the corresponding V_i', *relevant* if the weight of V_i' is n. In that case V_i' contains some, but not all of the $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$, and no other vector of weight less than or equal to n. (Each relevant V_i' cannot contain all of the vectors $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$, or else, being a convex filter, it would also contain the vector $\bar{\delta}$. This would contradict our assumption that $\bar{\delta} \notin V'$.)

Suppose that $\bar{w}_j \in U_j$, $j \in [m]$, have minimum weight. Then for every relevant $t_i' = \sum_{j \in [m]} c_{ij} x_j$, it holds that $\sum_{j \in [m]} c_{ij} \bar{w}_j$ has weight n and thus must be in $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$ (Lemma 35). Hence, no c_{ij} can be greater than 2, and there cannot be two coefficients equal to 2. The same fact holds for the coefficients in the linear term t. Thus, two cases arise.

Case 1: $t = x_1 + \cdots + x_m$. Since $\bar{\delta} \in V$ and V is a proper subset of Γ, by Lemma 35 there exist vectors $\bar{w}_j \in U_j$, $j \in [m]$, with $\bar{\delta} = \bar{w}_1 + \cdots + \bar{w}_m$, i.e. for each $i \in [n]$ there is a unique $j \in [m]$ such that the ith component of $\bar{w}_j$ is 1, and the ith component of any other $\bar{w}_k$ is 0. Since the operations are monotonic, we may also assume that $U_j = [[\bar{w}_j]]$, for all $j \in [m]$. Indeed, we have that, in $\mathbf{B}_n$,

$$\bar{\delta} \in t([[\bar{w}_1]], \ldots, [[\bar{w}_m]])$$

and

$$\bar{\delta} \notin t'([[\bar{w}_1]], \ldots, [[\bar{w}_m]]),$$

since $t'([[\bar{w}_1]],\ldots,[[\bar{w}_m]]) \subseteq t'(U_1,\ldots,U_m) = \Delta$. But if $t'([[\bar{w}_1]],\ldots,[[\bar{w}_m]])$ is not Δ, then it is in $CF(\mathbb{N}^n)$, and we may infer that

$$t([[\bar{w}_1]],\ldots,[[\bar{w}_m]]) \subseteq t'([[\bar{w}_1]],\ldots,[[\bar{w}_m]]),$$

contradicting the fact that $\bar{\delta}$ is contained in $t([[\bar{w}_1]],\ldots,[[\bar{w}_m]])$, but not in $t'([[\bar{w}_1]],\ldots, [[\bar{w}_m]])$. Thus, $t'([[\bar{w}_1]],\ldots,[[\bar{w}_m]]) = \Delta$.

Assume that $\bar{w}_1$ has two or more components equal to 1, say the first and the second components are 1. Since $\bar{w}_1 + \cdots + \bar{w}_m = \bar{\delta}$, the first two components of $\bar{w}_2,\ldots,\bar{w}_m$ are 0. Furthermore, as $\bar{\gamma}_1 \in \Gamma = t'(U_1,\ldots,U_m)$ in $CF(\mathbb{N}^n)$, there is a relevant t'_i with $\bar{\gamma}_1 \in V'_i$. This is possible only if $\bar{\gamma}_1 = \sum_{j\in[m]} c_{ij}\bar{w}_j$. But in that case the coefficient c_{i1} is 2. Thus, the first two components of $\bar{\gamma}_1$ would be 2, which is contradiction. Hence $\bar{w}_1$ has a single component equal to 1. In the same way, each of the vectors $\bar{w}_j$ is a unit vector, and $m = n$ follows, contrary to our assumption that $m < n$.

Case 2: $t = 2x_1 + x_2 + \cdots + x_m$. In this case, we may assume that there exist $\bar{w}_1,\ldots,\bar{w}_m$ and $\bar{v}_1$ with $\bar{v}_1 + \bar{w}_1 + \cdots + \bar{w}_m = \bar{\delta}$, $U_1 = [[\bar{w}_1,\bar{v}_1]]$ and $U_j = [[\bar{w}_j]]$, for $j \geqslant 2$. Note that $\bar{w}_1$ and $\bar{v}_1$ have equal weight, for otherwise V would contain a vector of weight strictly smaller than n. Again, we can conclude that $\bar{v}_1$ and each $\bar{w}_j$ have exactly one non-zero component, which is a 1. Using this, a contradiction is easily reached. Suppose that the first component of $\bar{w}_1$ is 1, say. Then there must be some i such that $t'_i = 2x_1 + x_2 + \cdots + x_m$. But then $t \geqslant t'$ holds in $\mathbf{B}_n$. $\square$

We are now ready to prove Theorem 32.

Proof of Theorem 32. Given an integer $n \geqslant 3$, consider the algebra $\mathbf{B}_n$ and the simple inequation $r_n \geqslant s_n$, where the terms r_n and s_n were defined below Eq. (10). For each $i \in [n]$, let $\bar{p}_i$ denote the ith n-dimensional unit vector whose components are all 0 except for a 1 in the ith position. We have

$$r_n([[\bar{p}_1]],\ldots,[[\bar{p}_n]]) = [[\bar{\delta}]]$$

and

$$s_n([[\bar{p}_1]],\ldots,[[\bar{p}_n]]) = \Delta$$

in $\mathbf{B}_n$. Thus $\mathbf{B}_n \not\models r_n \geqslant s_n$, i.e., $\mathbf{B}_n \not\models e_n^\wedge$. On the other hand $e_n^\wedge$ holds in $\mathscr{V}(\mathbf{N}_\wedge)$, and moreover, by Proposition 45, $\mathbf{B}_n$ satisfies all identities in at most $n - 1$ variables that hold in $\mathscr{V}(\mathbf{N}_\wedge)$. Hence, the collection of identities in at most $n - 1$ variables that hold in $\mathscr{V}(\mathbf{N}_\wedge)$ does not prove $e_n^\wedge$, and thus is not a basis for $\mathscr{V}(\mathbf{N}_\wedge)$. $\square$

Remark 46. The model construction upon which the proof of Theorem 32 is based is similar in spirit to the one we used in [1] to show that the variety $\mathscr{V}(\mathbf{N}_\vee)$ is not finitely based. The proof of that result was based upon the realization that the equations e_n below hold in $\mathbf{N}_\vee$, and in fact in $\mathbf{Z}_\vee$, for each $n \geqslant 2$:

$$e_n: \quad r_n \vee q_n = q_n, \tag{14}$$

where r_n was defined below (10) and

$$
\begin{aligned}
q_n = {} & (2x_1 + x_3 + x_4 + \cdots + x_{n-1} + x_n) \\
& \vee (x_1 + 2x_2 + x_4 + \cdots + x_{n-1} + x_n) \\
& \vdots \\
& \vee (x_1 + x_2 + x_3 + \cdots + x_{n-2} + 2x_{n-1}) \\
& \vee (x_2 + x_3 + x_4 + \cdots + x_{n-1} + 2x_n).
\end{aligned}
$$

In [1] we constructed a sequence of ciw-semirings $\mathbf{A}_n$ ($n \geqslant 2$) which satisfy all the equations in at most $n - 1$ variables that hold in $\mathcal{V}(\mathbf{N}_\vee)$, but in which the equation e_n fails. This shows that the collection of equations in at most $n - 1$ variables that hold in $\mathcal{V}(\mathbf{N}_\vee)$ does not prove e_n, and thus is not a basis of identities for $\mathcal{V}(\mathbf{N}_\vee)$. Unlike the algebras $\mathbf{B}_n$, the ciw-semirings $\mathbf{A}_n$ are finite, and consist of non-empty convex ideals contained in $[(\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}]]$, together with $[(\{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}]] - \{\bar{\delta}\}$, and an extra element $\top$.

The aforementioned results from [1] lead to an alternative proof of Theorem 30. Indeed, assume that there is a natural number $n \geqslant 2$ such that the collection E_n of equations in at most n variables that hold in $\mathcal{V}(\mathbf{Z}_\vee)$ is a basis for it. Since the equation e_{n+1} holds in $\mathcal{V}(\mathbf{Z}_\vee)$, it follows that E_n proves e_{n+1}. However, this contradicts the aforementioned results from [1]. In fact, since $\mathbf{N}_\vee$ is a subalgebra of $\mathbf{Z}_\vee$, the equations in E_n also hold in $\mathbf{N}_\vee$, and, thus, cannot prove e_{n+1}.

Remark 47. The alternative proof of Theorem 30 mentioned in the above remark is, in fact, applicable in a rather general setting. For the sake of this generalization, which is meant to apply to algebras whose addition operation need be neither commutative nor associative, we rephrase the family of equations in (14) as follows:

$$e'_n: \quad r'_n \vee q'_n = q'_n \quad (n \geqslant 2), \tag{15}$$

where

$$
\begin{aligned}
r'_n = {} & x_1 + (x_2 + (\cdots + x_n) \cdots) \quad \text{and} \\
q'_n = {} & (x_1 + (x_1 + (x_3 + (x_4 + (\cdots + (x_{n-1} + x_n) \cdots))))) \\
& \vee (x_1 + (x_2 + (x_2 + (x_4 + (\cdots + (x_{n-1} + x_n) \cdots))))) \\
& \vdots \\
& \vee (x_1 + (x_2 + (x_3 + (\cdots + (x_{n-2} + (x_{n-1} + x_{n-1})) \cdots)))) \\
& \vee (x_2 + (x_3 + (x_4 + (\cdots + (x_{n-1} + (x_n + x_n)) \cdots)))).
\end{aligned}
$$

(The join subterms in q'_n can be arbitrarily parenthesized.) The reader will find it easy to rephrase the family of equations $e_n^\wedge$ given in (10) in similar fashion.

The argument used in the previous remark can be used to show, *mutatis mutandis*, that:

Theorem 48. *Let $\mathcal{V}$ be any variety that contains $\mathbf{N}_\vee$ (respectively, $\mathbf{N}_\wedge$), and in which e'_n (resp., the rephrasing of $e^\wedge_n$) holds for every $n \geqslant 2$. Then $\mathcal{V}$ has no axiomatization in a bounded number of variables.*

Suppose that $\mathcal{V}$ is generated by the algebra $\mathbf{A}$. Then, the proviso of the above statement is met if the following conditions hold:
(1) $\mathbf{N}_\vee$ *(respectively, $\mathbf{N}_\wedge$) embeds in $\mathbf{A}$,*
(2) *the $\vee$ (respectively, $\wedge$) operation on $\mathbf{A}$ is a semilattice operation,*
(3) $\mathbf{A}$ *is linearly ordered by the semilattice order, and the $+$ operation is monotonic.*
An application of Theorem 48 *will be presented in Section* 6.4.

Remark 49. Since the algebra $\mathbf{N}_\vee^-$ is isomorphic to $\mathbf{N}_\wedge$, Theorems 31 and 32 apply equally well to it.

We have presented several examples of non-finitely based ciw-semirings. However, all of the ciw-semirings that we have studied so far are infinite. This prompts us to formulate the following:

Problem. Let $\mathbf{A}$ be a finite ciw-semiring. Is the variety $\mathcal{V}(\mathbf{A})$ finitely based?

4. Tropical semirings

Our aim in this section will be to investigate the equational theories of the tropical semirings studied in the literature that are obtained by adding bottom elements to the ciw-semirings presented in the previous section. More specifically, we shall study the following semirings:

$$\mathbf{Z}_{\vee,-\infty} = (\mathbb{Z} \cup \{-\infty\}, \vee, +, -\infty, 0),$$
$$\mathbf{N}_{\vee,-\infty} = (\mathbb{N} \cup \{-\infty\}, \vee, +, -\infty, 0)$$

and

$$\mathbf{N}_{\vee,-\infty}^- = (\mathbb{N}^- \cup \{-\infty\}, \vee, +, -\infty, 0).$$

Since $\mathbf{Z}_{\vee,-\infty}$, $\mathbf{N}_{\vee,-\infty}$ and $\mathbf{N}_{\vee,-\infty}^-$ are isomorphic to the semirings:

$$\mathbf{Z}_{\wedge,\infty} = (\mathbb{Z} \cup \{\infty\}, \wedge, +, \infty, 0),$$
$$\mathbf{N}_{\wedge,\infty}^- = (\mathbb{N}^- \cup \{\infty\}, \wedge, +, \infty, 0)$$

and

$$\mathbf{N}_{\wedge,\infty} = (\mathbb{N} \cup \{\infty\}, \wedge, +, \infty, 0),$$

respectively, the results that we shall obtain apply equally well to these algebras.

The semirings $\mathbf{Z}_{\wedge,\infty}$ and $\mathbf{N}_{\wedge,\infty}$ are usually referred to as the *equatorial semiring* [22] and the *tropical semiring* [35], respectively. The semiring $\mathbf{N}_{\vee,-\infty}$ is called the *polar semiring* in [24].

Our study of the equational theories of these algebras will proceed as follows. First, we shall offer some general facts relating the equational theory of a ciw-semiring $\mathbf{A}$ to the theory of the ci-semiring $\mathbf{A}_\perp$ defined in Section 2. In particular, in Section 4.1 we shall provide a necessary and sufficient condition that ensures that $\mathbf{A}$ and $\mathbf{A}_\perp$ satisfy the same equations in the language of ciw-semirings, and a necessary and sufficient condition ensuring the validity of a simple inequation in $\mathbf{A}_\perp$ for positive ciw-semirings $\mathbf{A}$. We use these conditions to relate the non-finite axiomatizability of $\mathscr{V}(\mathbf{A}_\perp)$ to the non-finite axiomatizability of $\mathscr{V}(\mathbf{A})$. Then, in Section 4.2, we shall apply our general study to derive the facts that all tropical semirings have exponential time decidable, but non-finitely based equational theories. Our general results, together with those proven in Section 3, will also give geometric characterizations of the valid equations in the tropical semirings $\mathbf{Z}_{\vee,-\infty}$ and $\mathbf{N}^-_{\vee,-\infty}$, and thus in $\mathbf{Z}_{\wedge,\infty}$ and $\mathbf{N}_{\wedge,\infty}$, but not in $\mathbf{N}_{\vee,-\infty}$, or in the isomorphic semiring $\mathbf{N}^-_{\wedge,\infty}$. The task of providing a geometric description of the valid equations for these semirings will be accomplished in Section 4.3, where we shall also show that $\mathscr{V}(\mathbf{N}_{\vee,-\infty})$ can be axiomatized over $\mathscr{V}(\mathbf{Z}_{\vee,-\infty})$ by a single equation.

4.1. Adding $\perp$

In Section 2, we saw how to generate a ci-semiring $\mathbf{A}_\perp = (A_\perp, \vee, +, \perp, 0)$ from any ciw-semiring $\mathbf{A} = (A, \vee, +, 0)$ by freely adding a bottom element $\perp$ to it. We now go on to study some general relationships between the equational theories of these two structures. Our investigations will proceed as follows. We shall first offer a result to the effect that if $\mathbf{A}$ is a ciw-semiring satisfying a certain technical property, then the variety it generates has a finite axiomatization (or an axiomatization in a bounded number of variables) if, and only if, so does $\mathscr{V}(\mathbf{A}_\perp)$ (cf. Theorem 55). Positive ciw-semirings, however, do not afford the technical property mentioned in the statement of Theorem 55. For this reason, we shall then prove two theorems that will allow us to lift results pertaining to the non-existence of finite axiomatizations, and of axiomatizations in a bounded number of variables, for non-trivial positive ciw-semirings to the free ci-semirings they generate (cf. Theorems 62 and 63). The developments of this section will be applied in Section 4.2 to obtain decidability and non-finite axiomatizability results for the tropical semirings associated with the ciw-semirings we studied in Section 3.

Recall that a simple inequation in the variables $x_1, \dots, x_n$ is of the form

$$t \leqslant \bigvee_{i \in [k]} t_i,$$

where $k > 0$, and t and the t_i ($i \in [k]$) are linear combinations of the variables $x_1, \dots, x_n$.

As mentioned above, our order of business will, first of all, be to study the relationships between the equational theories of ciw-semirings satisfying a certain technical property and those of the free ci-semirings they generate. The following definition introduces the notions that we shall use in the formulation of the aforementioned technical property.

Definition 50. A simple inequation $t \leqslant t'$ is called *non-expansive* if every variable that occurs in t' also occurs in t.

Suppose that $t \leqslant t'$ is a simple inequation. We say that $t \leqslant t'$ *has a kernel*, or that the *kernel of $t \leqslant t'$ exists*, if t' contains at least one linear subterm all of whose variables appear in t. Moreover, in this case we say that the kernel of $t \leqslant t'$ is the simple inequation $t \leqslant t''$, where the linear terms of t'' are those linear terms of t' whose variables all appear in t.

Thus, if $t \leqslant t'$ is non-expansive, then its kernel is the inequation $t \leqslant t'$. Note that the kernel of a simple inequation that holds in a ciw-semiring **A** need not hold in **A**. For example, the simple inequation $x \leqslant (x + y) \vee 0$ holds in every positive ciw-semiring, but its kernel $x \leqslant 0$ only holds in trivial positive semirings.

Example 51. The kernel of $x \leqslant x \vee y$ is $x \leqslant x$. The inequations $0 \leqslant x$ and $x \leqslant x + y \vee x + z$ have no kernel.

Lemma 52. *Suppose that* **A** *is a ciw-semiring. Then a simple inequation holds in* $\mathbf{A}_\perp$ *iff it has a kernel that holds in* **A**. *Thus, if an inequation is non-expansive, then it holds in* **A** *iff it holds in* $\mathbf{A}_\perp$.

Proof. Suppose that $t \leqslant t'$ is a simple inequation. If it has no kernel, then each linear subterm of t' contains a variable not occurring in t. Assign 0 to the variables that occur in t, and $\perp$ to any other variable. It follows that t evaluates to 0 while t' evaluates to $\perp$, proving that $t \leqslant t'$ does not hold in $\mathbf{A}_\perp$. Assume now that $t \leqslant t'$ has kernel $t \leqslant t''$. If $\mathbf{A} \not\models t \leqslant t''$ then for some evaluation in A of the variables appearing in t we have that the value of t in the algebra **A**, denoted a, is not less than, or equal to, the value b of t''. Assign $\perp$ to all other variables appearing in t'. Since in $\mathbf{A}_\perp$ term t evaluates to a and t' evaluates to b, it follows that $t \leqslant t'$ does not hold in $\mathbf{A}_\perp$. On the other hand, if $\mathbf{A} \models t \leqslant t''$, then $\mathbf{A}_\perp \models t \leqslant t'$. Indeed, this holds true when t evaluates to $\perp$. Assume that t evaluates to an element of A, the carrier set of **A**. Then the value of each variable occurring in t belongs to the set A. Thus, since each variable of t'' appears in t and since $t \leqslant t''$ holds in **A**, we have that the value of t is less than, or equal to, the value of t'', which in turn is less than, or equal to, the value of t'. $\square$

The above lemma has a number of useful corollaries relating the equational theory of a ciw-semiring with that of the free ci-semiring it generates.

Corollary 53. *The following conditions are equivalent for a ciw-semiring* **A**:
(1) *Every simple inequation that holds in* **A** *also holds in* $\mathbf{A}_\perp$.
(2) *Every equation in the language of ciw-semirings that holds in* **A** *also holds in* $\mathbf{A}_\perp$.
(3) **A** *and* $\mathbf{A}_\perp$ *satisfy the same equations in the language of ciw-semirings.*
(4) *For each simple inequation that holds in* **A**, *the kernel of the inequation exists and holds in* **A**.

(5) *There exists a set E of non-expansive simple inequations such that $E_{ciw} \cup E$ is an axiomatization of $\mathscr{V}(\mathbf{A})$.*

When these conditions hold, $\mathscr{V}(\mathbf{A}_\perp)$ is axiomatized by $E_{ci} \cup E$, where E is given as above.

Corollary 54. *Suppose that each inequation that holds in $\mathbf{A}$ has a kernel that holds in $\mathbf{A}$. Then $\mathscr{V}(\mathbf{A}_\perp)$ is axiomatized over $\mathscr{V}(\mathbf{A})$ by the equations*

$$x \vee \perp = x, \tag{16}$$

$$x + \perp = \perp. \tag{17}$$

We are now in a position to prove the promised result that will allow us to lift (non-)finite axiomatizability results from certain ciw-semirings to the free ci-semirings they generate.

Theorem 55. *Suppose that each inequation that holds in $\mathbf{A}$ has a kernel that holds in $\mathbf{A}$. Then $\mathscr{V}(\mathbf{A})$ has a finite axiomatization iff $\mathscr{V}(\mathbf{A}_\perp)$ has. Moreover, $\mathscr{V}(\mathbf{A})$ has an axiomatization by equations in a bounded number of variables iff $\mathscr{V}(\mathbf{A}_\perp)$ has.*

Proof. We first prove that $\mathscr{V}(\mathbf{A})$ has a finite axiomatization iff so does $\mathscr{V}(\mathbf{A}_\perp)$. One direction is obvious from Corollary 54. Suppose now that $\mathscr{V}(\mathbf{A}_\perp)$ has a finite axiomatization. Then, by Corollary 54 and the compactness theorem, there is a finite set E of simple equations that hold in $\mathbf{A}$ such that $E_{ci} \cup E$ is an axiomatization of $\mathscr{V}(\mathbf{A}_\perp)$. Since simple inequations that hold in $\mathbf{A}$ have kernels that hold in $\mathbf{A}$, we may assume that each inequation in E is non-expansive. We claim that $E_{ciw} \cup E$ is an axiomatization of $\mathscr{V}(\mathbf{A})$. Indeed, all of the equations in $E_{ciw} \cup E$ hold in $\mathbf{A}$. Assume that there is an equation $t = t'$ that holds in $\mathbf{A}$ but fails in a model $\mathbf{B}$ of the set of equations $E_{ciw} \cup E$. Then consider the ci-semiring $\mathbf{B}_\perp$. By Lemma 52, it satisfies each simple inequation in E, so that $\mathbf{B}_\perp$ is a model of $E_{ci} \cup E$. But since $t = t'$ fails in $\mathbf{B}$, it also fails in $\mathbf{B}_\perp$, contradicting the fact that $E_{ci} \cup E$ is an axiomatization of $\mathscr{V}(\mathbf{A})$. Thus, $E_{ciw} \cup E$ proves each equation that holds in $\mathbf{A}$, so that $\mathscr{V}(\mathbf{A})$ has a finite axiomatization. A similar reasoning proves that if $\mathscr{V}(\mathbf{A}_\perp)$ has an equational axiomatization in a bounded number of variables, then $\mathscr{V}(\mathbf{A})$ also has such an axiomatization. Indeed, suppose that $\mathscr{V}(\mathbf{A}_\perp)$ has an axiomatization in a bounded number of variables, say n. Since every simple inequation that holds in $\mathbf{A}$ has a kernel that holds in $\mathbf{A}$, by Corollary 53 we may assume, furthermore, that this axiomatization is given by the axioms of E_{ci} and a collection E of non-expansive simple inequations such that $E_{ciw} \cup E$ axiomatizes $\mathscr{V}(\mathbf{A})$. It follows that $\mathscr{V}(\mathbf{A})$ also has an axiomatization in a bounded number of variables. $\square$

We now turn to positive ciw-semirings. Note, first of all, that not every inequation that holds in a positive ciw-semiring has a kernel. For instance, as remarked in Example 51, the defining inequation for these structures, viz. $0 \leqslant x$, has no kernel. Thus the above theorem cannot be used for positive ciw-semirings. Our aim in the remainder of this section will be to prove statements that will allow us to lift negative results on

the axiomatizability of non-trivial positive ciw-semirings to the free ci-semirings they generate.

Definition 56. An equation is regular if its two sides contain the same variables (cf. eg., [38]). An inequation is regular if every variable contained in the left-hand side of the inequation also appears on the right-hand side. An inequation is strictly regular if its two sides contain the same variables.

Thus, a simple inequation is strictly regular iff it is regular and non-expansive.

Lemma 57. *Suppose that* $\mathbf{A}$ *is a non-trivial positive ciw-semiring and* $t \leqslant t'$ *is a simple inequation that holds in* $\mathbf{A}$. *Then* $t \leqslant t'$ *is regular, i.e., every variable that appears in* t *also appears in* t'.

Proof. Assume to the contrary that t contains a variable x which does not appear in t'. Then, substituting 0 for all other variables occurring in t or t', we obtain that $\mathbf{A}$ satisfies the inequation $nx \leqslant 0$, where n denotes the non-zero coefficient of x in t. Since $\mathbf{A}$ is positive it satisfies $0 \leqslant x$ and hence $x \leqslant nx$. It follows that $x = 0$ holds in $\mathbf{A}$, contradicting the assumption that $\mathbf{A}$ is non-trivial. □

Lemma 58. *Suppose that* $\mathbf{A}$ *is a non-trivial positive ciw-semiring. Then for each simple inequation* $t \leqslant t'$ *that holds in* $\mathbf{A}$ *there is a strictly regular inequation* $t \leqslant t''$ *that also holds in* $\mathbf{A}$ *such that* $E_{\mathrm{ciw}}^+ \cup \{t \leqslant t''\}$ *proves* $t \leqslant t'$.

Proof. Let t'' be the simple term that results from t' by substituting 0 for each variable that does not occur in t. □

We call the non-expansive, in fact strictly regular, inequation $t \leqslant t''$ constructed above the *projection* of $t \leqslant t'$.

Lemma 59. *Suppose that* $\mathbf{A}$ *is a non-trivial positive ciw-semiring. Then a simple inequation holds in* $\mathbf{A}_\perp$ *iff it has a strictly regular kernel that holds in* $\mathbf{A}$.

Proof. This follows from Lemmas 52 and 57. □

The two lemmas above are the key to the proof of the following result, which offers axiomatizations for the variety generated by a non-trivial positive ciw-semiring and for that generated by its associated free ci-semiring.

Corollary 60. *Suppose that* $\mathbf{A}$ *is a non-trivial positive ciw-semiring. Then* $\mathscr{V}(\mathbf{A})$ *is axiomatized by the set of equations* $E_{\mathrm{ciw}}^+ \cup E$, *where* E *denotes the set of all regular equations that hold in* $\mathbf{A}$. *Moreover,* $\mathscr{V}(\mathbf{A}_\perp)$ *is axiomatized by* $E_{\mathrm{ci}} \cup E$.

Proof. For the first part, we need to show that whenever an algebra $\mathbf{B}$ satisfies all the equations in $E_{\mathrm{ciw}}^+ \cup E$, then $\mathbf{B}$ satisfies any equation that holds in $\mathbf{A}$. But, with respect to

E_{ciw}, any equation is equivalent to a finite set of simple inequations (see Corollary 8). Moreover, by Lemma 58, for every simple inequation $t \leqslant t'$ that holds in $\mathbf{A}$ there is a strictly regular inequation $t \leqslant t''$ that also holds in $\mathbf{A}$ such that $E_{\mathrm{ciw}}^{+} \cup \{t \leqslant t''\}$ proves $t \leqslant t'$. Thus, since all such inequations hold in $\mathbf{B}$, it follows that $\mathbf{B}$ satisfies any equation that holds in $\mathbf{A}$.

For the second claim, observe that, by Lemma 59, any equation in E holds in $\mathbf{A}_\perp$, as does any equation in E_{ci}. Suppose now that an algebra $\mathbf{B}$ satisfies all the equations in $E_{\mathrm{ci}} \cup E$. By Corollary 8, we only need to show that $\mathbf{B}$ satisfies any simple inequation that holds in $\mathbf{A}_\perp$. Lemma 59 tells us that these are the simple inequations that have a strictly regular kernel that holds in $\mathbf{A}$. But these inequations can be proven from those in E, and thus hold in $\mathbf{B}$. $\square$

Remark 61. If $\mathbf{A}$ is a trivial ciw-semiring, then $\mathscr{V}(\mathbf{A})$ is axiomatized by the single equation $x = y$. However, this equation is not provable from $E_{\mathrm{ciw}}^{+} \cup E$, where E denotes the set of all regular equations that hold in $\mathbf{A}$. Thus the assumption of non-triviality in the statement of the above result is necessary.

Theorem 62. *Suppose that* $\mathbf{A}$ *is a non-trivial positive ciw-semiring. If the variety* $\mathscr{V}(\mathbf{A}_\perp)$ *has a finite axiomatization then so does* $\mathscr{V}(\mathbf{A})$.

Proof. Let E denote the set of strictly regular, simple inequations that hold in $\mathbf{A}$. By Corollary 60, $E_{\mathrm{ci}} \cup E$ proves all the equations satisfied by $\mathbf{A}_\perp$. Suppose that $\mathscr{V}(\mathbf{A}_\perp)$ is finitely axiomatizable. Then, by the compactness theorem, there is a finite set $F \subseteq E$ such that the set of equational axioms $E_{\mathrm{ci}} \cup F$ forms a complete axiomatization of $\mathscr{V}(\mathbf{A}_\perp)$. We claim that $E_{\mathrm{ciw}}^{+} \cup F$ is a complete axiomatization of the variety $\mathscr{V}(\mathbf{A})$. Indeed, all of the equations in this set hold in $\mathbf{A}$. Moreover, if $E_{\mathrm{ciw}}^{+} \cup F$ does not form a complete set of equations for $\mathscr{V}(\mathbf{A})$, then, by Corollary 60, there are some algebra $\mathbf{B}$ and a simple inequation $t \leqslant t'$ in E such that $\mathbf{B}$ satisfies all of the equations in $E_{\mathrm{ciw}}^{+} \cup F$, but such that $t \leqslant t'$ fails in $\mathbf{B}$. Consider the algebra $\mathbf{B}_\perp$. By Lemma 1, $\mathbf{B}_\perp$ satisfies the equations in E_{ci}, and, by Lemma 59, $\mathbf{B}_\perp$ satisfies the equations in F. Since $t \leqslant t'$ does not hold in $\mathbf{B}$, it follows that $t \leqslant t'$ does not hold in $\mathbf{B}_\perp$. But $t \leq t'$ is in E and thus holds in $\mathbf{A}_\perp$, so that $E_{\mathrm{ci}} \cup F$ is not a complete set of identities for $\mathscr{V}(\mathbf{A}_\perp)$. $\square$

Theorem 63. *Suppose that* $\mathbf{A}$ *is a non-trivial positive ciw-semiring. If the variety* $\mathscr{V}(\mathbf{A}_\perp)$ *has an equational axiomatization in a bounded number of variables, then so has* $\mathscr{V}(\mathbf{A})$.

Proof. Assume that the set E of valid equations of $\mathbf{A}_\perp$ in at most $n \geqslant 3$ variables forms an equational axiomatization of $\mathscr{V}(\mathbf{A}_\perp)$. With respect to the axioms E_{ci}, each equation in E may be transformed into a finite set of simple inequations in at most n variables. Any such inequation holds in $\mathbf{A}$ and is thus regular by Lemma 57. Now, using the equations in E_{ciw}^{+}, each inequation may be replaced by its projection which also has at most n variables. Any such simple inequation is strictly regular. Let E' denote the resulting set of inequations. By construction, we have that $E_{\mathrm{ci}} \cup E'$ is an axiomatization of $\mathscr{V}(\mathbf{A}_\perp)$. Moreover, since $n \geqslant 3$, each equation in this set has at most n variables.

As in the proof of the preceding theorem, we can show that $E_{\text{ciw}}^{+} \cup E'$ is a complete set of axioms for $\mathscr{V}(\mathbf{A})$. $\quad\square$

4.2. Decidability and non-finite axiomatizability

We now apply the results of the previous subsection to show that each of the tropical semirings defined thus far has a decidable but non-finitely based equational theory.

Proposition 64. *Every simple inequation that holds in* $\mathbf{N}_{\vee}^{-}$ *(respectively, in* $\mathbf{Z}_{\vee}$*) has a kernel that holds in* $\mathbf{N}_{\vee}^{-}$ *(resp., in* $\mathbf{Z}_{\vee}$*).*

Proof. Assume to the contrary that the simple inequation $t \leqslant t'$ holds in $\mathbf{N}_{\vee}^{-}$ but does not have a kernel. Then each linear subterm of t' contains a variable that does not appear in t. If we assign -1 to all such variables and 0 to any other variable then t evaluates to 0 while t' evaluates to a negative number. Thus $t \leqslant t'$ fails in $\mathbf{N}_{\vee}^{-}$, contradicting our assumption. Since $\mathbf{N}_{\vee}^{-}$ is a subalgebra of $\mathbf{Z}_{\vee}$, every simple inequation that holds in $\mathbf{Z}_{\vee}$ also has a kernel.

We now show that the kernel $t \leqslant t''$ of a simple inequation $t \leqslant t'$ that holds in $\mathbf{N}_{\vee}^{-}$ also holds in $\mathbf{N}_{\vee}^{-}$. To this end, assume that $t \leqslant t''$ fails in $\mathbf{N}_{\vee}^{-}$ and that $t' = t'' \vee u$, with u such that each of its linear subterms contains a variable not occurring in t. Let $\bar{x}$ be the vector of variables occurring in t, and $\bar{y}$ be the vector of variables occurring in u but not in t. Since $t \leqslant t''$ fails in $\mathbf{N}_{\vee}^{-}$, there is a vector $\bar{a}$ of non-positive integers such that $t(\bar{a})$, the result of evaluating t with respect to $\bar{a}$, is greater than $t''(\bar{a})$. Evaluate u with respect to the assignment that extends $\bar{a}$ by mapping each variable in $\bar{y}$ to $t''(\bar{a})$. The resulting value of u is no greater than $t''(\bar{a})$, showing that $t \leqslant t'$ fails in $\mathbf{N}_{\vee}^{-}$.

A similar argument can be used to show that the kernels of simple inequations that hold in $\mathbf{Z}_{\vee}$ also hold in $\mathbf{Z}_{\vee}$. $\quad\square$

Remark 65. In fact, the kernel of any simple inequation that holds in $\mathbf{Z}_{\vee}$ is strictly regular.

Theorem 66. *For each of the tropical semirings* $\mathbf{A} = \mathbf{Z}_{\vee,-\infty}, \mathbf{N}_{\vee,-\infty}, \mathbf{N}_{\vee,-\infty}^{-}$, *the equational theory of* $\mathscr{V}(\mathbf{A})$ *is decidable in exponential time. Suppose that* $\bar{d} \leqslant U$ *is a simple inequation. Then* $\bar{d} \leqslant U$ *holds in* $\mathscr{V}(\mathbf{Z}_{\vee,-\infty})$ *if and only if* $\bar{d} \in [U]$ *and it holds in* $\mathscr{V}(\mathbf{N}_{\vee,-\infty}^{-})$ *if and only if* $\bar{d} \in [[U]]$.

Proof. This follows from Proposition 64, the corresponding facts for the ciw-semirings $\mathbf{Z}_{\vee}, \mathbf{N}_{\vee} \ \mathbf{N}_{\vee}^{-}$ and from Lemmas 52 and 59. $\quad\square$

Remark 67. A simple inequation $\bar{d} \leqslant U$ holds in $\mathscr{V}(\mathbf{Z}_{\vee}(-\infty))$ if and only if $\bar{d} \in [U]$. (We recall that, for every ciw-semiring $\mathbf{A}$, we write $\mathbf{A}(\bot)$ for the algebra $(A_{\bot}, \vee, +, 0)$ obtained by adding $\bot$ to the carrier set of $\mathbf{A}$, but not to the signature.) Therefore the structures $\mathbf{Z}_{\vee}$, $\mathbf{Z}_{\vee}(-\infty)$ and $\mathbf{Z}_{\vee,-\infty}$ satisfy the same equations in the language of ciw-semirings.

A geometric description of the equations that hold in $\mathscr{V}(\mathbf{N}_{\vee,-\infty})$ will be given in Section 4.3.

Theorem 68. *For each of the tropical semirings* $\mathbf{A} = \mathbf{Z}_{\vee,-\infty}, \mathbf{N}_{\vee,-\infty}, \mathbf{N}_{\vee,-\infty}^-$, *the variety* $\mathscr{V}(\mathbf{A})$ *has no axiomatization in a bounded number of variables.*

Proof. This follows immediately from the corresponding facts for the ciw-semirings $\mathbf{Z}_\vee, \mathbf{N}_\vee$ $\mathbf{N}_\vee^-$ and Theorems 55 and 63. $\quad\square$

Since $\mathbf{Z}_{\wedge,\infty}$ and $\mathbf{N}_{\wedge,\infty}$ are isomorphic to $\mathbf{Z}_{\vee,-\infty}$ and $\mathbf{N}_{\vee,-\infty}^-$, respectively, Theorems 66 and 68 also apply to these semirings.

4.3. The algebra $\mathbf{N}_\vee(-\infty)$

We now study in some more detail the structure $\mathbf{N}_{\vee,-\infty}$, and in particular its ciw-semiring reduct $\mathbf{N}_\vee(-\infty)$ obtained by forgetting about the constant $\perp$. For this algebra, we shall offer concrete descriptions of the free algebras in the variety it generates, characterize the simple inequations that hold in it, and obtain an axiomatization for it relative to $\mathbf{Z}_\vee$.

Given a vector $\bar{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$, let $\mathbf{nz}(\bar{c})$ denote the set of all integers $i \in [n]$ with $c_i \neq 0$. (That is, $\mathbf{nz}(\bar{c})$ is the set of the non-zero positions in the vector $\bar{c}$.) When $U \subseteq \mathbb{N}^n$ we define:

$$\mathbf{nz}(U) = \bigcup \{\mathbf{nz}(\bar{c}) \colon \ \bar{c} \in U\}.$$

The following proposition is a reformulation of Lemma 52 that will be useful in the technical developments to follow.

Proposition 69. *Let* $\bar{d}$ *be a vector in* $\mathbb{N}^n$ *and* $U \subseteq \mathbb{N}^n$ *be non-empty and finite. The following are equivalent for a simple inequation* $\bar{d} \leqslant U$ *and a ciw-semiring* $\mathbf{A} = (A, \vee, +, 0)$.
(1) $\mathbf{A}_\perp \models \bar{d} \leqslant U$.
(2) *There exists a non-empty* $U' \subseteq U$ *such that* $\mathbf{nz}(U') \subseteq \mathbf{nz}(\bar{d})$ *and* $\mathbf{A} \models \bar{d} \leqslant U'$.
(3) $U_{\bar{d}} = \{\bar{c} \colon \ \bar{c} \in U \ and \ \mathbf{nz}(\bar{c}) \subseteq \mathbf{nz}(\bar{d})\}$ *is non-empty and* $\mathbf{A} \models \bar{d} \leqslant U_{\bar{d}}$.

The characterization of the simple inequations that hold in $\mathscr{V}(\mathbf{N}_\vee(-\infty))$, the variety generated by $\mathbf{N}_\vee(-\infty)$, will be based on a variation on the notion of convex ideal introduced in Definition 10. Note that, unlike $\mathbf{Z}_\vee$ and $\mathbf{Z}_\vee(-\infty)$ (cf. Remark 67), the ciw-semirings $\mathbf{N}_\vee$ and $\mathbf{N}_\vee(-\infty)$ do *not* have the same equational theory. For instance, the inequation

$$0 \leqslant x \tag{18}$$

holds in $\mathbf{N}_\vee$, but fails in $\mathbf{N}_\vee(-\infty)$. This observation, together with our characterization of the simple inequations that hold in $\mathbf{N}_\vee$, motivates the following definition.

Definition 70. Call a set $U \subseteq \mathbb{N}^n$ a *positive convex ideal* if for all non-empty $\{\bar{c}_1, \ldots, \bar{c}_k\} = U'$ included in U and $\bar{d} \in \mathbb{N}^n$, if $\mathbf{nz}(\bar{d}) \supseteq \mathbf{nz}(U')$ and there exist $\lambda_1, \ldots, \lambda_k \geqslant 0$, $\sum_{i \in [k]} \lambda_i = 1$ with $\bar{d} \leqslant \sum_{i \in [k]} \lambda_i \bar{c}_i$, then $\bar{d} \in U$.

Remark 71. In the above definition, we may replace the condition $\mathbf{nz}(\bar{d}) \supseteq \mathbf{nz}(U')$ by $\mathbf{nz}(\bar{d}) = \mathbf{nz}(U')$.

Proposition 72. *The intersection of any family of positive convex ideals is a positive convex ideal.*

Thus, each $U \subseteq \mathbb{N}^n$ is included in a least positive convex ideal $[(U]]^+$. For example, the positive convex ideal $[(\bar{p}_i]]^+$ ($i \in [n]$) generated by the ith unit vector only contains $\bar{p}_i$.

Lemma 73. *Let $U \subseteq \mathbb{N}^n$ and $\bar{d} \in \mathbb{N}^n$. Then $\bar{d} \in [(U]]^+$ if and only if there exists a non-empty $U' = \{\bar{c}_1, \ldots, \bar{c}_k\} \subseteq U$ such that $\mathbf{nz}(U') \subseteq \mathbf{nz}(\bar{d})$, and $\bar{d} \in [(U']]$.*

As an immediate corollary of the above lemma, we obtain a characterization of the simple inequations that hold in the algebra $\mathbf{N}_\vee(-\infty)$. Using this characterization and the general strategy employed in the proof of Corollary 20, we also immediately have that the equational theory of this algebra is decidable.

Corollary 74. *Suppose that U is a non-empty finite set in $\mathbb{N}^n$ and $\bar{d} \in \mathbb{N}^n$. Then*:
(1) *$\bar{d} \leqslant U$ holds in $\mathbf{N}_\vee(-\infty)$ iff $\bar{d} \in [(U]]^+$.*
(2) *There exists an algorithm to decide whether an equation holds in the structure $\mathbf{N}_\vee(-\infty)$.*

Proof. We only present a proof of the first claim. Assume that $\bar{d} \leqslant U$ holds in $\mathbf{N}_\vee(-\infty)$. Since the algebras $\mathbf{N}_\vee(-\infty)$ and $\mathbf{N}_{\vee,-\infty}$ have the same carrier set, this means that $\bar{d} \leqslant U$ also holds in $\mathbf{N}_{\vee,-\infty}$. By Proposition 69, there exists a non-empty $U' \subseteq U$ such that $\mathbf{nz}(U') \subseteq \mathbf{nz}(\bar{d})$ and $\mathbf{N}_\vee \models \bar{d} \leqslant U'$. By the first claim in Proposition 19, we have that $\bar{d} \in [(U']]$. Since $\mathbf{nz}(U') \subseteq \mathbf{nz}(\bar{d})$, Lemma 73 yields that $\bar{d} \in [(U]]^+$, which was to be shown. The proof of the converse implication is similar, and is therefore omitted. $\square$

We now offer concrete descriptions of the free algebras in the variety generated by $\mathbf{N}_\vee(-\infty)$.

Let $CI^+(\mathbb{N}^n)$ denote the set of all finite non-empty positive convex ideals in $\mathbb{N}^n$. We turn this set into a ciw-semiring. Suppose that $U, V \in CI^+(\mathbb{N}^n)$. We define

$$U \vee V = [(U \cup V]]^+,$$
$$U + V = [(U \oplus V]]^+,$$
$$0 = [(\bar{0}]]^+ = \{\bar{0}\}.$$

Following the structure of the proof of similar statements in Section 3.3, it is not too hard to show that:

Theorem 75. *For each $n \geqslant 0$, the algebra $CI^+(\mathbb{N}^n)$ is a ciw-semiring, and is freely generated in $\mathscr{V}(\mathbf{N}_\vee(-\infty))$ by the sets $[(\bar{p}_i)]^+$, $i \in [n]$.*

Our next aim will be to offer finite axiomatizations of the variety generated by $\mathbf{N}_\vee(-\infty)$ relative to those for $\mathscr{V}(\mathbf{Z}_\vee)$. This result will also yield finite axiomatizations of $\mathscr{V}(\mathbf{N}_\vee)$ relative to $\mathscr{V}(\mathbf{N}_\vee(-\infty))$. As a corollary, we shall obtain that $\mathscr{V}(\mathbf{N}_\vee(-\infty))$ is not finitely based.

Lemma 76. *The convex hull of any non-empty set U is included in $[(U)]^+$.*

Let $U \subseteq \mathbb{N}^n$. Define the *positive ideal* generated by U thus:

$$(U]^+ = \{\bar{d}: \exists \bar{c} \in U \text{ such that } \mathbf{nz}(\bar{c}) = \mathbf{nz}(\bar{d}) \text{ and } \bar{d} \leqslant \bar{c}\}.$$

The following technical result is the crux of the relative axiomatization results to follow.

Proposition 77. *Suppose that U is a non-empty subset of $\mathbb{N}^n$. Then $[(U]]^+$ is the convex hull of $(U]^+$.*

Proof. The proof is similar to that of the first statement in Proposition 13, but we present it in full as an aid to the reader.

Since $(U]^+ \subseteq [(U]]^+$, it follows by Lemma 76 that the convex hull of $(U]^+$ is included in $[(U]]^+$. In order to prove the other direction, suppose that

$$\bar{d} \leqslant \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k$$

for some $k > 0$, $\bar{c}_1, \ldots, \bar{c}_k \in U$ and $\lambda_1, \ldots, \lambda_k \geqslant 0$ with $\sum_{i \in [k]} \lambda_i = 1$. Moreover, assume that $\mathbf{nz}(\bar{d}) = \mathbf{nz}(\{\bar{c}_1, \ldots, \bar{c}_k\})$. By Lemma 73, it suffices to show that $\bar{d}$ is in the convex hull of the set $(V]^+$, where $V = \{\bar{c}_1, \ldots, \bar{c}_k\}$. We shall prove this by induction on

$$r = k + n + w,$$

where we use w to denote the sum of the weights of the vectors $\bar{c}_i$ ($i \in [k]$), i.e., the sum of all of their entries.

The base case is when $r = 2$. Then $n = k = 1$ and $\bar{d} = \bar{c} = \bar{0}$, so that our claim holds.

For the inductive step, suppose that $r > 2$. We proceed with the proof by distinguishing three cases.

Case 1: If there exists some $j \in [n]$ with $d_j = 0$, then for this j, we have $c_{ij} = 0$ for all $i \in [k]$. We can then remove the jth components of all the vectors to obtain $\bar{d}'$ and $\bar{c}'_1, \ldots, \bar{c}'_k$ of dimension $n-1$ with $\bar{d}' \leqslant \lambda_1 \bar{c}'_1 + \cdots + \lambda_k \bar{c}'_k$ and $\mathbf{nz}(\bar{d}') = \mathbf{nz}(\{\bar{c}'_1, \ldots, \bar{c}'_k\})$. Let $W = \{\bar{c}'_1, \ldots, \bar{c}'_k\}$. Thus, by induction, $\bar{d}'$ is in the convex hull of $(W]^+$, so that $\bar{c}$ is in the convex hull of $(V]^+$. The case that $n = 1$ is handled separately: we have $\bar{d} = \bar{c}_1 = \cdots = \bar{c}_k = \bar{0}$, and our claim is trivial.

Case 2: If there exists an $i \in [k]$ with $\lambda_i = 0$, then it follows from the inductive hypothesis that $\bar{d}$ is already in the convex hull of the set $(W)^+$, where $W = V - \{\bar{c}_i\}$.

Case 3: If none of the previous two cases applies, then $d_j > 0$ for all $j \in [n]$ and $\lambda_i > 0$ for all $i \in [k]$. Suppose that there exists some j such that $c_{ij} = 1$ for all $i \in [k]$. Then also $d_j = 1$ and we may remove the jth components of the vectors to obtain $\bar{d}'$ and $\bar{c}_i'$, $i \in [k]$, as before. Using the inductive hypothesis, it follows as in case 1 above that $\bar{d}$ is in the convex hull of $(V)^+$. The case that $n = 1$ is again handled separately.

Suppose now that for each j there is some i_j with $c_{i_j j} > 1$. Let $\bar{e} = \lambda_1 \bar{c}_1 + \cdots + \lambda_k \bar{c}_k$. If for some j

$$
\begin{aligned}
\bar{d} &\leqslant \lambda_1 \bar{c}_1 + \cdots + \lambda_{i_j-1} \bar{c}_{i_j-1} + \lambda_{i_j}(\bar{c}_{i_j} - \bar{p}_j) + \lambda_{i_j+1} \bar{c}_{i_j+1} + \cdots + \lambda_k \bar{c}_k \\
&= \bar{e} - \lambda_{i_j} \bar{p}_j,
\end{aligned}
$$

then, by induction, $\bar{d}$ is contained in the convex hull of $(W)^+$, where W is the set $\{\bar{c}_1, \ldots, \bar{c}_{i_j-1}, \bar{c}_{i_j} - \bar{p}_j, \bar{c}_{i_j+1}, \ldots, \bar{c}_k\} \subseteq (V)^+$. It follows that $\bar{d}$ is in the convex hull of $(V)^+$. Otherwise, we have that

$$
e_j - \lambda_{i_j} \leqslant d_j \leqslant e_j,
$$

for all j. But this means that $\bar{d}$ is inside the n-dimensional cube determined by the vectors

$$
\bar{v}_K = \bar{e} - \sum_{j \in K} \lambda_{i_j} \bar{p}_j,
$$

where K ranges over all subsets of $[k]$. Since these vectors $\bar{v}_K$ are all in the convex hull of $(V)^+$, it follows that $\bar{d}$ belongs to the convex hull of $(V)^+$, which was to be shown. $\square$

Corollary 78. *A simple inequation $\bar{d} \leqslant U$ holds in $\mathbf{N}_\vee(-\infty)$ iff $\bar{d}$ is in the convex hull of $(U)^+$ iff the simple inequation $\bar{d} \leqslant (U)^+$ holds in $\mathbf{Z}_\vee$.*

Using the above results, and following the lines of the proof of Theorem 28, we can now show the promised results on the relative axiomatization of the varieties $\mathscr{V}(\mathbf{N}_\vee(-\infty))$ and $\mathscr{V}(\mathbf{N}_\vee)$.

Corollary 79. $\mathscr{V}(\mathbf{N}_\vee(-\infty))$ *can be axiomatized over* $\mathscr{V}(\mathbf{Z}_\vee)$ *by the inequation*

$$
x \leqslant x + x. \tag{19}
$$

Furthermore $\mathscr{V}(\mathbf{N}_{\vee,-\infty})$ *can be axiomatized over* $\mathscr{V}(\mathbf{Z}_{\vee,-\infty})$ *by the above inequation.*

Corollary 80. $\mathscr{V}(\mathbf{N}_\vee)$ *can be axiomatized over* $\mathscr{V}(\mathbf{N}_\vee(-\infty))$, *or over* $\mathscr{V}(\mathbf{Z}_\vee)$, *by the inequation* (18), *i.e.,* $0 \leqslant x$.

Using the aforementioned relative axiomatization results, and following the lines of the proof of Theorem 30, we have that:

Corollary 81. *The variety $\mathscr{V}(\mathbf{N}_\vee(-\infty))$ is not finitely based. Moreover, there exists no n such that the set of all equations E_n in at most n variables that hold in $\mathbf{N}_\vee(-\infty)$ is a complete axiomatization of $\mathscr{V}(\mathbf{N}_\vee(-\infty))$.*

5. Adding more constants

We now proceed our investigations of the equational theories of structures based upon ciw-semirings by investigating the effect of adding a top element $\top$ to them.

5.1. Adding $\top$

Suppose that $\mathbf{A} = (A, \vee, +, 0)$ is any algebra and $\top \notin A$. We define

$$a + \top = \top + a = \top,$$
$$a \vee \top = \top \vee a = \top$$

for all $a \in A_\top = A \cup \{\top\}$. Below we shall consider $A_\top$ equipped with the $\vee$ and $+$ operations, extended as above, and the constant 0. Consistently with previous notation, we write $\mathbf{A}_\top$ if $\top$ is added to the signature of the resulting algebra, and $\mathbf{A}(\top)$ for the structure resulting by adding $\top$ only to the carrier set. When $\mathbf{A}$ is one of the ci(w)-semirings $\mathbf{Z}_\vee, \mathbf{N}_\vee$, etc., then we also write $\mathbf{A}_\infty$ and $\mathbf{A}(\infty)$ for $\mathbf{A}_\top$ and $\mathbf{A}(\top)$, respectively.

Proposition 82. *Any equation that holds in $\mathbf{A}(\top)$ also holds in $\mathbf{A}$. Moreover, $\mathbf{A}(\top)$ satisfies an equation iff the equation is regular and holds in $\mathbf{A}$.*

Proof. Since $\mathbf{A}$ is a subalgebra of $\mathbf{A}(\top)$, the first claim is obvious. As for the second claim, assume that $t = t'$ holds in $\mathbf{A}$ and t and t' contain the same variables, say $x_1, \ldots, x_n$. Since $t = t'$ holds in $\mathbf{A}$, the only way that $t = t'$ may fail in $\mathbf{A}(\top)$ is that there exist some $a_1, \ldots, a_n$ such that at least one of the a_i is $\top$ and $t(a_1, \ldots, a_n) \neq t'(a_1, \ldots, a_n)$. But in that case both sides are $\top$.

On the other hand, if t' contains a variable that does not appear in t, say, and if the variables of t and t' are $x_1, \ldots, x_n$, then let $a_i \in A$ whenever x_i appears in t, and let $a_i = \top$ otherwise. We have that $t(a_1, \ldots, a_n) \in A$ but $t'(a_1, \ldots, a_n) = \top$. Hence the equation $t = t'$ fails in $\mathbf{A}(\top)$. $\quad\square$

Corollary 83. *The algebras $\mathbf{A}$ and $\mathbf{A}(\top)$ satisfy the same regular equations. Moreover, $\mathbf{A}$ and $\mathbf{A}(\top)$ satisfy the same equations iff every equation that holds in $\mathbf{A}$ is regular.*

Corollary 84. *$\mathbf{A}$ is a ciw-semiring iff so is $\mathbf{A}(\top)$. Suppose that $\mathbf{A}$ is a ciw-semiring and that every simple inequation that holds in $\mathbf{A}$ is regular. Then $\mathbf{A}$ and $\mathbf{A}(\top)$ satisfy the same equations.*

Proof. The first fact follows by Proposition 82, noting that every defining equation of ciw-semirings is regular. The second fact follows by noting that the following two conditions are equivalent for a ciw-semiring **A**:
(1) Every equation that holds in **A** is regular.
(2) Every simple inequation that holds in **A** is regular. □

Remark 85. The same facts hold if we consider algebras containing more constants such as algebras $(A, \vee, +, \bot, 0)$ equipped with a constant $\bot$.

We might wish to add the symbol $\top$ also to the signature and consider equations involving $\top$ that hold in $\mathbf{A}_\top$. The following proposition gives a characterization of the equations of this kind that hold in $\mathbf{A}_\top$ relative to those that hold in **A**.

Proposition 86. *For any algebra* $\mathbf{A} = (A, \vee, +, 0)$ *or* $\mathbf{A} = (A, \vee, +, \bot, 0)$, *an equation between terms possibly involving* $\top$ *holds in* $\mathbf{A}_\top$ *iff*
• *either both sides contain an occurrence of* $\top$;
• *or neither of them does, the equation holds in* **A** *and is regular.*

The simple proof is omitted. The above result has a number of useful corollaries relating the equational theories of $\mathbf{A}_\top$ and **A**.

Corollary 87. *If the equational theory of* **A** *is decidable in time* $O(t(n))$, *with* $t(n) \geqslant n^2$, *then so is the equational theory of* $\mathbf{A}_\top$.

Corollary 88. *Suppose that* $\mathbf{A} = (A, \vee, +, 0)$ *or* $\mathbf{A} = (A, \vee, +, \bot, 0)$ *is a given algebra and E is a set of equations possibly involving* $\top$. *Let* E_0 *denote the set of equations in E not containing* $\top$. *Then E, together with the equations*

$$x + \top = \top, \tag{20}$$

$$x \vee \top = \top, \tag{21}$$

is an equational basis for $\mathbf{A}_\top$ *iff* E_0 *is a basis for the regular equations* [2] *that hold in* **A**. *In particular, if all equations satisfied by* **A** *are regular, then E together with* (20) *and* (21) *is an equational basis for* $\mathbf{A}_\top$ *iff* E_0 *is an equational basis for* **A**.

Corollary 89. *Suppose that* **A** *is a ci(w)-semiring such that every simple inequation that holds in* **A** *is regular. Let E denote a set of equations and define* E_0 *as in the statement of Corollary* 88. *Then E, together with* (20) *and* (21), *is an equational basis for* $\mathbf{A}_\top$ *iff* E_0 *is an equational basis for* **A**.

---

[2] This means that E_0 consists of regular equations that hold in **A**. Moreover, E_0 proves all regular equations that hold in **A**.

Corollary 90. *Suppose that* **A** *is a ci(w)-semiring such that every simple inequation that holds in* **A** *is regular. Then* **A** *has a finite basis for its identities iff so does* $\mathbf{A}_\top$. *Moreover,* **A** *has an axiomatization in a bounded number of variables iff so does* $\mathbf{A}_\top$.

5.2. Applications

We now proceed to apply the general results developed in Section 5.1 to the algebras obtained by adding top elements to some of the ciw-semirings we study in this paper.

Proposition 91. *Let* **A** *be any of the commutative idempotent (weak) semi-rings* $\mathbf{Z}_\vee$, $\mathbf{N}_\vee$, $\mathbf{Z}_{\vee,-\infty}$, $\mathbf{N}_{\vee,-\infty}$, $\mathbf{Z}_\vee(-\infty)$ *and* $\mathbf{N}_\vee(-\infty)$. *Then* $\mathbf{A}(\infty)$ *and* $\mathbf{A}_\infty$ *are not finitely based, and have no axiomatization in a bounded number of variables. Moreover, the equational theory of* $\mathbf{A}_\infty$ *is decidable in exponential time.*

Proof. Immediate from the preceding results, the fact that each of $\mathbf{Z}_\vee$, $\mathbf{N}_\vee$, $\mathbf{Z}_\vee(-\infty)$, $\mathbf{N}_\vee(-\infty)$, $\mathbf{Z}_{\vee,-\infty}$ and $\mathbf{N}_{\vee,-\infty}$ is a ci(w)-semiring satisfying only regular simple inequations, and from the results established in Sections 3 and 4. $\square$

6. Variations on tropical semirings

We now examine some variations on the tropical semirings studied so far in this paper. These include structures whose carrier sets are the (non-negative) rational or real numbers (Section 6.1), semirings whose product operation is standard multiplication (Section 6.2), and the semirings studied by Mascle and Leung in [30] and [26,27], respectively, (Section 6.3). We also offer results on the equational theory of some algebras based on the ordinals proposed by Mascle in [29] (Section 6.4).

6.1. Structures over the rational and real numbers

We now proceed to study tropical semirings over the (non-negative) rationals and reals, and their underlying ciw-semirings. More precisely, we shall consider the following ciw-semirings:

$$\mathbf{Q}_\vee = (\mathbb{Q}, \vee, +, 0), \quad \mathbf{Q}_\vee^+ = (\mathbb{Q}^+, \vee, +, 0), \quad \mathbf{Q}_\wedge^+ = (\mathbb{Q}^+, \wedge, +, 0),$$
$$\mathbf{R}_\vee = (\mathbb{R}, \vee, +, 0), \quad \mathbf{R}_\vee^+ = (\mathbb{R}^+, \vee, +, 0), \quad \mathbf{R}_\wedge^+ = (\mathbb{R}^+, \wedge, +, 0),$$

where $\mathbb{Q}^+$ and $\mathbb{R}^+$ denote the sets of non-negative rational and real numbers, respectively. We shall also investigate the tropical semirings associated with the aforementioned ciw-semirings, viz. the structures

$$\mathbf{Q}_{\vee,-\infty} = (\mathbb{Q} \cup \{-\infty\}, \vee, +, -\infty, 0), \quad \mathbf{Q}_{\vee,-\infty}^+ = (\mathbb{Q}^+ \cup \{-\infty\}, \vee, +, -\infty, 0),$$
$$\mathbf{R}_{\vee,-\infty} = (\mathbb{R} \cup \{-\infty\}, \vee, +, -\infty, 0), \quad \mathbf{R}_{\vee,-\infty}^+ = (\mathbb{R}^+ \cup \{-\infty\}, \vee, +, -\infty, 0),$$
$$\mathbf{Q}_{\wedge,\infty}^+ = (\mathbb{Q}^+ \cup \{\infty\}, \wedge, +, \infty, 0), \quad \mathbf{R}_{\wedge,\infty}^+ = (\mathbb{R}^+ \cup \{\infty\}, \wedge, +, \infty, 0).$$

We begin by noting the following result, to the effect that each of the algebras on a dense carrier set has the same equational theory of its corresponding discrete structure:

Lemma 92. The following statements hold:
(1) *The ciw-semirings* $\mathbf{Q}_\vee$, $\mathbf{R}_\vee$ *and* $\mathbf{Z}_\vee$ *have the same equational theory.*
(2) *The ciw-semirings* $\mathbf{Q}_\vee^+$, $\mathbf{R}_\vee^+$ *and* $\mathbf{N}_\vee$ *have the same equational theory.*
(3) *The ciw-semirings* $\mathbf{Q}_\wedge^+$, $\mathbf{R}_\wedge^+$ *and* $\mathbf{N}_\wedge$ *have the same equational theory.*
(4) *The ci-semirings* $\mathbf{Q}_{\vee,-\infty}$, $\mathbf{R}_{\vee,-\infty}$ *and* $\mathbf{Z}_{\vee,-\infty}$ *have the same equational theory.*
(5) *The ci-semirings* $\mathbf{Q}_{\vee,-\infty}^+$, $\mathbf{R}_{\vee,-\infty}^+$ *and* $\mathbf{N}_{\vee,-\infty}$ *have the same equational theory.*
(6) *The ci-semirings* $\mathbf{Q}_{\wedge,\infty}^+$, $\mathbf{R}_{\wedge,\infty}^+$ *and* $\mathbf{N}_{\wedge,\infty}$ *have the same equational theory.*

Proof. We only outline the proof of the first statement of the lemma. Since $\mathbf{Z}_\vee$ is a subalgebra of $\mathbf{Q}_\vee$, which is in turn a subalgebra of $\mathbf{R}_\vee$, it follows that the equational theory of $\mathbf{R}_\vee$ is included in that of $\mathbf{Q}_\vee$, which is in turn included in that of $\mathbf{Z}_\vee$. For the converse, assume that

$$t(x_1,\ldots,x_n) = t'(x_1,\ldots,x_n)$$

holds in $\mathbf{Z}_\vee$. Let $r_1,\ldots,r_n$ be rationals, and write $r_i = q_i/q$, where the q_i $(i \in [n])$ and q are integers, with q positive. Now

$$\begin{aligned} t(r_1,\ldots,r_n) &= \frac{t(q_1,\ldots,q_n)}{q} \\ &= \frac{t'(q_1,\ldots,q_n)}{q} \\ &= t'(r_1,\ldots,r_n). \end{aligned}$$

Thus, any valid equation of $\mathbf{Z}_\vee$ holds in $\mathbf{Q}_\vee$. The fact that any equation of $\mathbf{Q}_\vee$ holds in $\mathbf{R}_\vee$ follows from the continuity of the term functions. The proof of the other statements is similar. □

As an immediate corollary of the above lemma, and of the non-finite axiomatizability and decidability results presented earlier in the paper, we have that:

Theorem 93. *Let* $\mathbf{A}$ *be any of the algebras* $\mathbf{Q}_\vee$, $\mathbf{R}_\vee$, $\mathbf{Q}_\vee^+$, $\mathbf{R}_\vee^+$, $\mathbf{Q}_\wedge^+$, $\mathbf{R}_\wedge^+$, $\mathbf{Q}_{\vee,-\infty}$, $\mathbf{R}_{\vee,-\infty}$, $\mathbf{Q}_{\vee,-\infty}^+$, $\mathbf{R}_{\vee,-\infty}^+$, $\mathbf{Q}_{\wedge,\infty}^+$ *and* $\mathbf{R}_{\wedge,\infty}^+$. *Then*:
(1) *The variety* $\mathscr{V}(\mathbf{A})$ *is not finitely based.*
(2) *For every natural number n, the collection of equations in at most n variables that hold in* $\mathscr{V}(\mathbf{A})$ *is not a basis for its identities.*
(3) *The equational theory of* $\mathscr{V}(\mathbf{A})$ *is decidable in exponential time.*

Remark 94. The structure $\mathbf{R}_{\vee,-\infty} = (\mathbb{R}, \vee, +, -\infty, 0)$ is the well-known max-plus algebra, whose plethora of applications are discussed in, e.g., [15].

All of the algebras, whose carrier sets are the set of rational or real numbers, that we have discussed so far in this section sometimes appear in their isomorphic form

as min-plus algebras. As a further corollary of the above theorem, we therefore have that:

Corollary 95. *Let A be either $\mathbb{Q}$ or $\mathbb{R}$. Then the algebras $\mathbf{A}_\wedge = (A, \wedge, +, 0)$ and $\mathbf{A}_{\wedge,\infty} = (A \cup \{\infty\}, \wedge, +, \infty, 0)$ are not finitely based. Moreover, for every $n \in \mathbb{N}$, the collection of equations in at most n variables that hold in $\mathbf{A}_\wedge$ (respectively, $\mathbf{A}_{\wedge,\infty}$) does not form an equational basis for $\mathbf{A}_\wedge$ (resp., $\mathbf{A}_{\wedge,\infty}$). Finally, the equational theories of $\mathbf{A}_\wedge$ and $\mathbf{A}_{\wedge,\infty}$ are decidable in exponential time.*

6.2. Min-max-times algebras

We now apply the results we have previously obtained to the study of the equational theories of the ci(w)-semirings

$$\mathbf{A}_{\vee,\times} = (A \backslash \{0\}, \vee, \times, 1),$$
$$\mathbf{A}_{\wedge,\times} = (A \backslash \{0\}, \wedge, \times, 1),$$
$$\mathbf{A}_{\vee,\times,0} = (A, \vee, \times, 0, 1)$$

and

$$\mathbf{A}_{\wedge,\times,0} = (A, \wedge, \times, 0, 1),$$

where A is any of the sets $\mathbb{N}$, $\mathbb{Q}^+$ or $\mathbb{R}^+$, and $\times$ is standard multiplication.

Proposition 96. *Let A be any of the sets $\mathbb{N}$, $\mathbb{Q}^+$ or $\mathbb{R}^+$. Then*:
(1) $\mathscr{V}(\mathbf{A}_{\vee,\times}) = \mathscr{V}(\mathbf{A}_\vee)$,
(2) $\mathscr{V}(\mathbf{A}_{\wedge,\times}) = \mathscr{V}(\mathbf{A}_\wedge)$,
(3) $\mathscr{V}(\mathbf{A}_{\vee,\times,0}) = \mathscr{V}(\mathbf{A}_{\vee,-\infty})$ *and*
(4) $\mathscr{V}(\mathbf{A}_{\wedge,\times,0}) = \mathscr{V}(\mathbf{A}_{\wedge,-\infty})$.

Proof. We only show statement 1, when A is $\mathbb{N}$. The proof of the remaining claims is similar.

First of all, note that $\mathbf{N}_\vee$ is isomorphic to the subalgebra of $\mathbf{N}_{\vee,\times}$ determined by the natural numbers that are a power of 2. Conversely, observe that $\mathbf{N}_{\vee,\times}$ is isomorphic to the algebra

$$(\log(\mathbb{N} \backslash \{0\}), \vee, +, 0),$$

where we write $\log(\mathbb{N} \backslash \{0\})$ for the collection of logarithms in base 2 of the positive natural numbers, via the mapping

$$n \mapsto \log n.$$

The above mapping is injective, and the set $\log(\mathbb{N} \backslash \{0\})$ is closed under addition, in light of the well-known equation

$$\log(x \times y) = \log x + \log y.$$

Note, furthermore, that $\mathbf{N}_\vee$ is a subalgebra of $(\log(\mathbb{N}\setminus\{0\}), \vee, +, 0)$, which is itself a subalgebra of $\mathbf{R}_\vee^+$. By Lemma 92, these three algebras have the same equational theory, and thus generate the same variety. $\square$

As a corollary of the above proposition, and of the results that we have previously established for min-max-plus ci(w)-semirings, we obtain the following result:

Corollary 97. *Let A be any of the sets $\mathbb{N}$, $\mathbb{Q}^+$ or $\mathbb{R}^+$. Then*:
(1) *The varieties generated by the algebras $\mathbf{A}_{\vee,\times}$, $\mathbf{A}_{\wedge,\times}$, $\mathbf{A}_{\vee,\times,0}$ and $\mathbf{A}_{\wedge,\times,0}$ are not finitely based. Moreover none of these varieties has an axiomatization in a bounded number of variables.*
(2) *There exists an exponential time algorithm to decide whether an equation holds in the structures $\mathbf{A}_{\vee,\times}$, $\mathbf{A}_{\wedge,\times}$, $\mathbf{A}_{\vee,\times,0}$ and $\mathbf{A}_{\wedge,\times,0}$.*

6.3. Mascle's and Leung's semirings

We now study the equational theory of some semirings originally proposed by Mascle and Leung, and discussed in the survey paper [32].

6.3.1. Mascle's semiring
In [30], Mascle introduced the semiring

$$\mathbf{P}_{-\infty} = (\mathbb{N} \cup \{-\infty, \infty\}, \vee, +, -\infty, 0)$$

where the addition operation satisfies the identities

$$-\infty + x = x + (-\infty) = -\infty.$$

(See also the survey paper [32] for information on this and other semirings proposed by Mascle.) Note that the ci-semiring $\mathbf{P}_{-\infty}$ is different from the structure $\mathbf{N}_{\vee,-\infty}(\infty)$ we studied in Section 5.2. Indeed, in $\mathbf{N}_{\vee,-\infty}(\infty)$ it holds that

$$-\infty + \infty = \infty + (-\infty) = \infty.$$

Instead, it is the case that $\mathbf{P}_{-\infty}$ is obtained by freely adding $-\infty$ to the ciw-semiring $\mathbf{N}_\vee(\infty)$.

Theorem 98. *The variety $\mathscr{V}(\mathbf{P}_{-\infty})$ is not finitely based, and affords no axiomatization in a bounded number of variables.*

Proof. The variety generated by the positive ciw-semiring $\mathbf{N}_\vee(\infty)$ has no axiomatization in a bounded number of variables (Proposition 91). By Theorem 63, the same holds true of $\mathscr{V}(\mathbf{P}_{-\infty})$. $\square$

6.3.2. Leung's semiring
Leung [26,27] introduced and studied the semiring

$$\mathbf{M} = (\mathbb{N} \cup \{\omega, \infty\}, \wedge, +, \infty, 0),$$

where the minimum operation is defined with respect to the order

$$0 < 1 < 2 < \cdots < \omega < \infty,$$

and addition in the tropical semiring $\mathbf{N}_{\wedge,\infty}$ is completed by stipulating that

$$x + \omega = \omega + x = \max\{x, \omega\}.$$

Thus the carrier of $\mathbf{M}$ is just the ordinal $\omega + 2$. It is easy to see that $\mathbf{M}$ is a ci-semiring.

We shall now proceed to show that Leung's semiring is also not finitely based. To this end, we relate the equational theory of the tropical semiring $\mathbf{N}_{\vee,-\infty}^{-}$ to that of

$$\mathbf{M}^{-} = (\mathbb{N}^{-} \cup \{-\omega, -\infty\}, \vee, +, -\infty, 0),$$

which is isomorphic to $\mathbf{M}$.

Lemma 99. *Let $t \leqslant t'$ be a simple inequation. Then $t \leqslant t'$ holds in $\mathbf{N}_{\vee,-\infty}^{-}$ iff it holds in $\mathbf{M}^{-}$.*

Proof. Since $\mathbf{N}_{\vee,-\infty}^{-}$ is a subalgebra of $\mathbf{M}^{-}$, every simple inequation that holds in $\mathbf{M}^{-}$ also holds in $\mathbf{N}_{\vee,-\infty}^{-}$. Conversely, let $t \leqslant t'$ be a simple inequation that holds in $\mathbf{N}_{\vee,-\infty}^{-}$. By Lemma 52, $t \leqslant t'$ has a kernel that holds in $\mathbf{N}_{\vee}^{-}$. This kernel holds in $\mathbf{M}^{-}$ (again by Lemma 52), and so does $t \leqslant t'$. $\square$

As immediate corollary of the above result, we have that $\mathbf{N}_{\vee,-\infty}^{-}$ and $\mathbf{M}^{-}$ have the same equational theory. Since Leung's semiring $\mathbf{M}$ and $\mathbf{M}^{-}$ are isomorphic, using Theorem 68, we therefore have that:

Proposition 100. *The variety $\mathscr{V}(\mathbf{M})$ generated by the semiring $\mathbf{M}$ has no finite axiomatization, and no axiomatization in a bounded number of variables.*

6.4. Ordinals

In [29] (see also the survey paper [32]), Mascle proposed to study min-plus algebras whose carrier sets consist of the collection of ordinals strictly smaller than a given ordinal α. Our aim in this section is to offer results to the effect that these algebras do not afford any axiomatization in a bounded number of variables. We begin by giving the precise definition of these structures.

Recall that each ordinal α can be represented as the well-ordered set of all the ordinals strictly smaller than it. When α is a power of ω, the first infinite ordinal, this set is closed under ordinal addition, giving rise to the structures

$$\alpha_{\vee} = (\alpha, \vee, +, 0)$$

and

$$\alpha_{\wedge} = (\alpha, \wedge, +, 0),$$

where $\vee$ and $\wedge$ denote the maximum and minimum operation over ordinals, respectively. Since ordinal addition is *not* commutative, these structures are not ciw-semirings unless $\alpha = 1$ or $\alpha = \omega$. However, except for commutativity of addition, they satisfy all the defining equations of ciw-semirings. Moreover, when $\alpha \neq 1$, the algebra $\alpha_\vee$ contains $\mathbf{N}_\vee$ as a subalgebra, and $\alpha_\wedge$ contains $\mathbf{N}_\wedge$. In both $\alpha_\vee$ and $\alpha_\wedge$, the carrier set α is linearly ordered by the semilattice order, and the $+$ operation is monotonic (see, e.g., [20]). Thus, by Theorem 48, we have that:

Theorem 101. *Suppose that $\alpha \neq 1$ is a power of ω. Then $\mathscr{V}(\alpha_\vee)$ cannot be axiomatized by equations in a bounded number of variables. The same fact holds for $\mathscr{V}(\alpha_\wedge)$.*

When α is an ordinal of the form ω^β, where β is itself a power of ω, the set α is closed under ordinal product, giving rise to the structures

$$\alpha_{\vee,\times} = (\alpha \backslash \{0\}, \vee, \times, 1),$$

and

$$\alpha_{\wedge,\times} = (\alpha \backslash \{0\}, \wedge, \times, 1).$$

Proposition 102. *The structures $\alpha_\vee$ and $\alpha_\wedge$ embed in $\omega^\alpha_{\vee,\times}$ and $\omega^\alpha_{\wedge,\times}$, respectively.*

Proof. The function mapping any ordinal $\beta < \alpha$ to ω^β is an embedding, in light of the equality

$$\omega^\beta \times \omega^\gamma = \omega^{\beta+\gamma},$$

and we are done. $\square$

From the above result, it follows that the algebra $\alpha_{\vee,\times}$ contains $\mathbf{N}_{\vee,\times}$ as a subalgebra, and $\alpha_{\wedge,\times}$ contains $\mathbf{N}_{\wedge,\times}$. In both $\alpha_{\vee,\times}$ and $\alpha_{\wedge,\times}$, the carrier set α is linearly ordered by the semilattice order, and the $\times$ operation is monotonic (see, e.g., [20]). Thus, by Theorem 48, we have that:

Theorem 103. *Suppose that α is an ordinal of the form ω^β, where β is itself a power of ω. Then $\mathscr{V}(\alpha_{\vee,\times})$ cannot be axiomatized by equations in a bounded number of variables. The same fact holds for $\mathscr{V}(\alpha_{\wedge,\times})$.*

Acknowledgements

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