THE EQUATIONAL THEORY OF KLEENE LATTICES

HAJNAL ANDRÉKA, SZABOLCS MIKULÁS, AND ISTVÁN NÉMETI

Abstract. Languages and families of binary relations are standard interpretations of Kleene algebras. It is known that the equational theories of these interpretations coincide and that the free Kleene algebra is representable both as a relational and as a language algebra. We investigate the identities valid in these interpretations when we expand the signature of Kleene algebras with the meet operation. In both cases meet is interpreted as intersection. We prove that in this case there are more identities valid in language algebras than in relational algebras (exactly three more in some sense), and representability of the free algebra holds for the relational interpretation but fails for the language interpretation. However, if we exclude the identity constant from the algebras when we add meet, then the equational theories of the relational and language interpretations remain the same, and the free algebra is representable as a language algebra, too. The moral is that only the identity constant behaves differently in the language and the relational interpretations, and only meet makes this visible.

Keywords: Kleene algebra, Kleene lattice, equational theory, language algebra, relation algebra

1. Introduction

Kleene algebras are extensively investigated in language theory and in programming logics, see, e.g., [3, 4, 5]. There are various definitions of Kleene algebras in the literature; following [5], by a Kleene algebra we mean an algebra satisfying a finite set of axioms (that we will not need in this paper). We will denote the class of Kleene algebras by KA.

The notation for the operations in a Kleene algebra is a bit problematic. Traditionally, we have $+$ (for addition or join), * (for the Kleene star), $\cdot$ (for multiplication or composition) and the constants 0 (for the additive identity) and 1 (for the multiplicative identity). However, since we want to add a full lattice structure, we will use $\cdot$ for meet and denote the multiplication by ; (following the relation algebra literature[1]) — this notation is also used in [6]. Hence the signature of Kleene algebras is $(+, ;, ^*, 0, 1)$, and Kleene algebras with meet, or *Kleene lattices*, have signature $(\cdot, +, ;, ^*, 0, 1)$. Another operation that

Corresponding author: Sz. Mikulás.

[1]Relation algebraists beware that 1 denotes the identity constant and not the top element. Also, the expression 'relation algebra' in this paper refers to algebras of binary relations, i.e., 'representable' algebras.

is sometimes added to Kleene algebras is converse (or conversion or inverse), we will denote it by $^\smile$. We denote by ω the set of natural numbers.

As usual in Kleene algebras, we may not mention multiplication ; explicitly and write $x ; y$ as xy. We will use the standard notation $x^0 := 1$, $x^{n+1} := x^n ; x$ and the abbreviation x^+ for $x ; x^* = x^* ; x$. We also note that $x^* = x^+ + 1$ is valid in KA, thus x^* implicitly uses the identity 1. Let KA^- denote the class of generalized subreducts of elements of KA to the signature $(+, ;, ^+, 0)$. That is, we omit 1 and replace * with $^+$. We call elements of KA^- *identity-free Kleene algebras.*

The two main types of Kleene algebras are language algebras and relation algebras. They are defined as follows. Let Σ be a set (alphabet) and Σ^* denote the free monoid of all finite words over Σ, including the empty word λ. The class of *language Kleene algebras* is defined as the class of subalgebras of algebras of the form

$$(\wp(\Sigma^*), +, ;, ^*, 0, 1)$$

where $+$ is set union, ; is complex concatenation (of words)

$$X ; Y = \{xy : x \in X, y \in Y\}, \tag{1.1}$$

* is the Kleene star operation

$$X^* = \{x_0 x_1 \dots x_{n-1} : n \in \omega, x_i \in X \text{ for each } i < n\}, \tag{1.2}$$

0 is the empty language and 1 is the singleton language consisting of the empty word λ. We will denote the class of language Kleene algebras by LKA. The class of *language Kleene lattices*, LKL, is defined analogously: subalgebras of $(\wp(\Sigma^*), \cdot, +, ;, ^*, 0, 1)$ where, in addition, $\cdot$ is interpreted as intersection.

The class of *relational Kleene algebras* is defined as the class of subalgebras of algebras of the form

$$(\wp(W), +, ;, ^*, 0, 1)$$

where $W = U \times U$ for some set U, $+$ is set union, ; is relation composition

$$X ; Y = \{(u, v) \in W : (u, w) \in X \text{ and } (w, v) \in Y \text{ for some } w\}, \tag{1.3}$$

* is reflexive-transitive closure, 0 is the emptyset and 1 is the identity relation restricted to W

$$1 = \{(u, v) \in W : u = v\}. \tag{1.4}$$

We will denote the class of relational Kleene algebras by RKA. The class of *relational Kleene lattices*, RKL, is defined analogously: again $\cdot$ is interpreted as intersection.

We will use similar notation for other similarity types: LKA^- and LKL^- denote language algebras, and RKA^- and RKL^- denote relation algebras of the similarity types where 1 and * are replaced by $^+$.

It is well known that the same equations are true in language Kleene algebras and in relational Kleene algebras:

$$\mathsf{Eq}(\mathsf{LKA}) = \mathsf{Eq}(\mathsf{RKA}) \tag{1.5}$$

where $\mathsf{Eq}(\mathsf{K})$ denotes the set of equations valid in the class K of algebras. In passing we note that they coincide with the equational theory $\mathsf{Eq}(\mathsf{KA})$ of Kleene algebras as well, see [5].

Equation (1.5) can be established by showing that LKA and RKA have the same free algebra, viz. the algebra of regular languages, see, e.g., [7] for the argument in the context of dynamic algebras.

One might wonder if we could prove more, e.g., whether the two classes of algebras coincide up to isomorphisms. If K is a class of algebras, then let $\mathbf{I}\mathsf{K}$ denote the class of algebras isomorphic to some element of K. It is the case that $\mathsf{LKA} \subseteq \mathbf{I}\mathsf{RKA}$. The proof relies on the following function (functor) assigning a binary relation to a language X over an alphabet Σ:

$$f(X) = \{(w, wx) : w \in \Sigma^* \text{ and } x \in X\}\,.$$

This function f, called the *Cayley representation*, respects the Kleene algebra operations: $+, ;, ^*, 0, 1$. Consequently, any language Kleene algebra is isomorphic to a relational Kleene algebra, and thus any equation valid in relational Kleene algebras is also valid in language Kleene algebras. The converse however fails: $\mathsf{RKA} \not\subseteq \mathbf{I}\mathsf{LKA}$. One trivial reason for this is that the identity in each language algebra is the one-element set $\{\lambda\}$, hence is an atom, while in relation algebras this is not so. This property is reflected in the following equational implications (i.e., quasi-equations) distinguishing LKA and RKA:

$$x \le 1 \to x \,;\, y = y \,;\, x \tag{1.6}$$

$$x \le 1 \to (x \,;\, y)^+ = x \,;\, (y^+) \tag{1.7}$$

where $x \le y$ abbreviates $y = x + y$.

The next question is whether then those relation algebras in which the identity is an atom, or whether identity-free relational Kleene algebras are isomorphic to language Kleene algebras. The answer here is in the negative, too. The following quasi-equation containing only the operation ; also distinguishes them:

$$\mathsf{LKA} \models x = x^3 \to x = x^2 \quad \text{while} \quad \mathsf{RKA} \not\models x = x^3 \to x = x^2\,. \tag{1.8}$$

Thus there are fewer language Kleene algebras (up to isomorphism) than relational Kleene algebras, and their quasi-equational theories are different, but their equational theories coincide.

The Cayley representation f preserves also meet. Consequently,

$$\mathsf{Eq}(\mathsf{RKL}) \subseteq \mathsf{Eq}(\mathsf{LKL})\,.$$

However, strict inclusion and not equality holds in this case. The reason is that the quasi-equations (1.6) and (1.7) can be translated to equations if we have both identity and meet. Indeed, we can replace x by $x \cdot 1$ in the consequent of the quasi-equations, e.g., (1.7) can be equivalently written as $((x \cdot 1) ; y)^+ = (x \cdot 1) ; (y^+)$ — see also the identities (3.2) and (3.3) in Section 3. One of our main theorems in this paper, Theorem 3.2, states that LKL can be axiomatized over RKL by these two equations plus one more (so, in a sense, there are only three equations valid in RKL which are not valid in LKL). The free language Kleene lattice is no longer representable as a language Kleene lattice, but it is representable as a relational Kleene lattice, see Theorem 3.1 and the remark following it.

The above quasi-equations and the proofs of the above-mentioned theorems all exploit that in language algebras the identity behaves very differently from the relational case: it is a one-element set $\{\lambda\}$, and it cannot be obtained as a concatenation of words distinct from λ. Indeed, if we omit occurrences of 1 (even implicitly as in x^*), then the equational theories of language and relation algebras coincide, and the free algebra is again representable as a language algebra, see Theorems 4.1 and 4.3.

What is the case with conversion? Well, the Cayley representation f does not preserve conversion, and indeed there are equations valid in relation algebras which are *not* valid in language algebras, e.g., $x \leq x ; x^{\smile} ; x$ is such. As one can see from this equation, the culprit is again the identity in a hidden way ($x ; x^{\smile}$ contains the domain $x ; x^{\smile} \cdot 1$ of x). In passing we note that if we do not have intersection, then the following axioms: Kleene algebra axioms, $(x + y)^{\smile} = x^{\smile} + y^{\smile}$, $(x ; y)^{\smile} = y^{\smile} ; x^{\smile}$, $(x^*)^{\smile} = (x^{\smile})^*$, $x^{\smile\smile} = x$ and $x \leq x ; x^{\smile} ; x$ axiomatize RKA with conversion, see [4]. In the present paper we state the main technical lemma (Lemma 2.5) for a similarity type containing converse, too, but in the rest of the paper we do not deal with converse.

2. Terms, graphs and words

In this section we consider the full signature $(\cdot, +, ;, ^*, 0, 1, ^{\smile})$ of Kleene lattices with conversion. $\mathsf{RKL}^{\smile}$ and $\mathsf{LKL}^{\smile}$ denote the classes of relational and language Kleene lattices with conversion, respectively. We recall a technique that allows us to concentrate on terms containing none of $+, ^*, 0$, and we introduce a graphic representation for these terms. Then we construct "characteristic" words to these terms, and we prove our key technical lemma, Lemma 2.5, which we will use in the subsequent sections.

2.1. **Ground terms and continuity.** A *ground term* is one in which neither of $+, ^*, 0$ occurs. (Note that variables can occur in ground terms.) First we define the set $\Gamma(\tau)$ of ground terms for any term τ, and then we show that τ can be replaced, in a sense, with $\Gamma(\tau)$. We use the notation $T \mathbin{\Delta} S = \{\tau \mathbin{\Delta} \sigma : \tau \in T, \sigma \in S\}$ for an operation Δ

and sets T and S of terms. For a variable x we let $\Gamma(x) = \{x\},\ \Gamma(0) = \emptyset,\ \Gamma(1) = \{1\}$,

$$\begin{aligned}
\Gamma(\tau + \sigma) &= \Gamma(\tau) \cup \Gamma(\sigma) \\
\Gamma(\tau \cdot \sigma) &= \Gamma(\tau) \cdot \Gamma(\sigma) \\
\Gamma(\tau \,;\, \sigma) &= \Gamma(\tau) \,;\, \Gamma(\sigma) \\
\Gamma(\tau^{\smile}) &= (\Gamma(\tau))^{\smile} \\
\Gamma(\tau^{*}) &= \bigcup\{\Gamma(\tau^{n}) : n \in \omega\}
\end{aligned}$$

and we let $GT = \bigcup_{\tau} \Gamma(\tau)$ denote the set of ground terms. For every term τ, let $\tau^{\mathfrak{A}}[k]$ denote the value of τ in the algebra $\mathfrak{A}$ under the evaluation k of variables in A. Now we show a *-continuity property allowing in many cases to concentrate on ground terms only.

Lemma 2.1. *For every term τ and language or relation algebra $\mathfrak{A}$ of signature* $(\cdot, +, ;, ^{*}, 0, 1, ^{\smile})$,

$$\tau^{\mathfrak{A}}[k] = \bigcup\{\sigma^{\mathfrak{A}}[k] : \sigma \in \Gamma(\tau)\}\,.$$

Proof. This is an easy induction on τ, by using complete additivity of $\cdot, ;, ^{\smile}$. □

Remark. The above lemma remains true if we replace * by $^+$, and in this case we have

$$\Gamma(\tau^{+}) = \bigcup\{\Gamma(\tau^{n}) : 1 \leq n \in \omega\}\,.$$

□

2.2. **Term graphs.** Term graphs are a graphic representation for ground terms. We recall the definition of term graphs; these were introduced in [1], but we use the notation of [2].

Let X be the set of our variables. An *X-labelled graph* (or simply just a labelled graph) is a structure $G = (V, E)$ where V is the set of vertices and $E \subseteq V \times X \times V$ is the set of labelled edges. Given two labelled graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a *homomorphism* $h\colon G_1 \to G_2$ is a map from V_1 to V_2 that preserves X-labelled edges: if $(u, x, v) \in E_1$, then $(h(u), x, h(v)) \in E_2$. Given an equivalence relation θ on V, the *quotient graph* is $G/\theta = (V/\theta, E/\theta)$ where V/θ is the set of equivalence classes of V and

$$E/\theta = \{(u/\theta, x, v/\theta) : (\exists u' \in u/\theta)(\exists v' \in v/\theta)(u', x, v') \in E\}\,.$$

A *2-pointed graph* is a labelled graph $G = (V, E)$ with two (not necessarily distinct) distinguished vertices $\iota, o \in V$. We will call ι the *input* and o the *output* vertex of G, respectively, and denote 2-pointed graphs as $G = (V, E, \iota, o)$. In the case of 2-pointed graphs, we require that a homomorphism preserves input and output vertices as well.

Let $G_1 \oplus G_2$ denote the disjoint union of G_1 and G_2. For 2-pointed graphs $G_1 = (V_1, E_1, \iota_1, o_1)$ and $G_2 = (V_2, E_2, \iota_2, o_2)$, we define their *composition* as

$$G_1 \,;\, G_2 = (((V_1, E_1) \oplus (V_2, E_2))/\theta, \iota_1/\theta, o_2/\theta)$$

where θ is the smallest equivalence relation on the disjoint union $V_1 \cup V_2$ that identifies o_1 with ι_2. The *meet* of G_1 and G_2 is defined as

$$G_1 \cdot G_2 = (((V_1, E_1) \oplus (V_2, E_2))/\theta, \iota_1/\theta, o_1/\theta)$$

where θ is the smallest equivalence relation on the disjoint union $V_1 \cup V_2$ that identifies ι_1 with ι_2 and o_1 with o_2. When no confusion is likely we will identify an equivalence class u/θ with u, hence ι_i/θ with ι_i and o_i/θ with o_i for $i \in \{1, 2\}$.

We define *term graphs* as special 2-pointed graphs by induction on the complexity of ground terms. Let

$$G(1) = (\{\iota\}, \emptyset, \iota, \iota)$$

i.e., in this case $\iota = o$. For variable x, we let

$$G(x) = (\{\iota, o\}, \{(\iota, x, o)\}, \iota, o)$$

where $\iota \neq o$. For terms σ and τ, we set

$$G(\sigma \cdot \tau) = G(\sigma) \cdot G(\tau) \text{ and } G(\sigma \,;\, \tau) = G(\sigma) \,;\, G(\tau)$$

while $G(\sigma^{\smile})$ is defined by swapping ι and o in $G(\sigma)$.

As an example consider the term $\tau = (x_1 \,;\, x_2{}^{\smile}) \cdot (y_1 \,;\, y_2)$. The graph $G(\tau)$ is drawn in Figure 1.

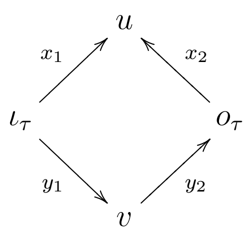

FIGURE 1. The graph $G(\tau)$ for $\tau = (x_1 \,;\, x_2{}^{\smile}) \cdot (y_1 \,;\, y_2)$

The next two lemmas are about the use of term graphs. Let τ be a ground term and $G(\tau) = (V(\tau), E(\tau), \iota_\tau, o_\tau)$. When $\mathfrak{A}$ is a language or relational Kleene lattice we will often omit the superscript $\mathfrak{A}$ and we will simply write $\tau[k]$. The reason is that in these cases k itself determines $\tau^{\mathfrak{A}}[k]$, this value is the same for all $\mathfrak{A}$ such that the values of k are in A. The following is proved in [1, Lemma 3]. We will not use it in this paper, we include it for seeing the analogy with the next lemma.

Lemma 2.2. *Let τ be a ground term, U be a set and k be an evaluation of the variables of τ in $\wp(U \times U)$. Then for every $(u, v) \in U \times U$, items* (1) *and* (2) *below are equivalent.*

(1) $(u, v) \in \tau[k]$.
(2) *There is a map $h\colon V(\tau) \to U$ such that $h(\iota_\tau) = u$, $h(o_\tau) = v$, and for every edge $(i, x, j) \in E(\tau)$, we have $(h(i), h(j)) \in k(x)$.*

By using Lemma 2.2 one can prove that $\mathsf{RKL}^{\smile} \models \tau \leq \sigma$ iff there is a homomorphism from $G(\sigma)$ to $G(\tau)$, see [1, Theorem 1].

We prove an analogous lemma for language algebras. For this we need to consider the natural *partial ordering on* $G(\tau)$ introduced in [2, Definition 4.6]. We briefly recall the definition. In $G(x)$ we have that $\iota_x \leq o_x$. In $G(\tau\,;\sigma)$ the ordering is the extension of the ones of $G(\tau)$ and $G(\sigma)$ by stipulating that each node in $G(\tau)$ precedes each node in $G(\sigma)$. The ordering in $G(\tau \cdot \sigma)$ is just the union of the ones of $G(\tau)$ and $G(\sigma)$. Finally, the ordering of $G(\tau^{\smile})$ is the reverse of that of $G(\tau)$. (In the notation of [2], we have that $u \leq v$ iff $\tau(u, v)$ is defined.) Notation: Let $w = w_1 \dots w_n$ be a word where $w_1, \dots, w_n$ are letters. Then $|w| = n$ denotes the length of w. Let $0 \leq i < j \leq n$. Then $w(i, j) = w_{i+1} \dots w_j$, $w(j, i) = w(i, j)^{\smile} = w_j \dots w_{i+1}$, and $w(j, j) = \lambda$ denote the corresponding subwords of w.

Lemma 2.3. *Let τ be a ground term, Σ be an alphabet and k be an evaluation of the variables of τ in $\wp(\Sigma^*)$. Then for every $w = w_1 \dots w_n \in \Sigma^*$, items* (1) *and* (2) *below are equivalent.*

(1) $w \in \tau[k]$.
(2) *There is an order-preserving map $h\colon V(\tau) \to n{+}1 = \{0, 1, \dots, n\}$ such that $h(\iota_\tau) = 0$, $h(o_\tau) = n$, and for every edge $(i, x, j) \in E(\tau)$, we have $w(h(i), h(j)) \in k(x)$.*

Proof. We proceed by induction on τ. All the cases are straightforward except perhaps one part of the case of composition where we need the ordering and the case of converse. We write out these parts of the proof.

Assume $h\colon V(\tau\,;\sigma) \to n + 1$ is an order-preserving map such that $h(\iota_{\tau;\sigma}) = 0$, $h(o_{\tau;\sigma}) = n$, and for every edge $(i, x, j) \in E(\tau)$, we have $w(h(i), h(j)) \in k(x)$. Let $q \in V(\tau\,;\sigma)$ be the vertex connecting $G(\tau)$ and $G(\sigma)$ in $G(\tau\,;\sigma)$, i.e., $q = o_\tau = \iota_\sigma$. Let $u = w(0, h(q))$, $v = w(h(q), n)$ and let g be defined as the restriction of h to $V(\tau)$. Then $g\colon V(\tau) \to |u| + 1$ by the order-preserving property of h. Also, g is order-preserving and satisfies the rest of the conditions for $G(\tau)$. By the induction hypothesis then $u \in \tau[k]$. Let now g be defined as $g(i) = h(i) - |u|$ for every $i \in V(\sigma)$. As before, $g\colon V(\sigma) \to |v| + 1$ satisfies the required conditions and so $v \in \sigma[k]$. Thus $w = uv \in \tau;\sigma[k]$, and we are done.

Assume now that $w \in \tau^{\smile}[k]$. We want to show the existence of an appropriate $h\colon V(\tau^{\smile}) \to n + 1$ where $n = |w|$. We have that $v \in \tau[k]$

where $v = v_1 \dots v_n = w_n \dots w_1$. Thus $v_i = w_{n+1-i}$ for $i \leq n$. By the induction hypothesis there is an appropriate $g\colon V(\tau) \to n+1$. We note that $V(\tau^{\smile}) = V(\tau)$ and $E(\tau^{\smile}) = E(\tau)$, just the ordering is the reverse and the endpoints are swapped. We define $h\colon V(\tau) \to n+1$ by $h(i) = n - g(i)$. Then h is order-preserving from $V(\tau^{\smile})$ and takes the endpoints $\iota_{\tau^{\smile}} = o_\tau$ and $o_{\tau^{\smile}} = \iota_\tau$ to 0 and n, respectively. Assume that $(i,x,j) \in E(\tau^{\smile})$. Then $v(g(i),g(j)) \in k(x)$ by the induction hypothesis. We show that $w(h(i),h(j)) = v(g(i),g(j))$. Assume that $h(i) < h(j)$. Then $g(i) > g(j)$. Now, $w(h(i),h(j)) = w_{h(i)+1} \dots w_{h(j)} = v_{n+1-h(i)-1} \dots v_{n+1-h(j)} = v_{g(i)} \dots v_{g(j)+1} = v(g(i),g(j))$. The other cases are completely analogous, and we are done. □

2.3. **Words associated to terms.** We now turn to the main construction and technical lemma for obtaining the results in this paper. To each identity-free ground term τ we will construct a word w_τ and a function $f_\tau\colon V(\tau) \to \omega$ which, in a sense, will be characteristic for τ, see Lemma 2.5. Instead of defining w_τ and f_τ formally, we just state the existence of these with the desired properties. From the proof of Lemma 2.4 one can extract an explicit construction for w_τ and f_τ, but we will not need these concrete forms.

Lemma 2.4. *For every identity-free ground term τ, there are a word $w_\tau = w_1 \dots w_n$ and a map $f_\tau\colon V(\tau) \to n+1$ such that the following conditions* (1)–(3) *hold.*

(1) *w_τ is repetition-free (i.e., $w_i \neq w_j$ for all $1 \leq i < j \leq n$).*
(2) *f_τ satisfies $f_\tau(\iota_\tau) = 0$, $f_\tau(o_\tau) = n$ and f_τ is order-preserving (i.e., if $u \leq v$ in $G(\tau)$, then $f_\tau(u) \leq f_\tau(v)$).*
(3) *f_τ is strongly injective in the sense that the values of f_τ are separated by at least two letters (i.e., $|f_\tau(u) - f_\tau(v)| \geq 2$ for all distinct $u, v \in G(\tau)$).*

Proof. Let τ be an identity-free ground term and consider its graph $G(\tau) = (V(\tau), E(\tau), \iota_\tau, o_\tau)$. Since τ is identity-free, we can extend the natural partial ordering on $V(\tau)$ to a linear ordering, also denoted by $\leq$, say, $\iota_\tau = q_0 \leq q_1 \leq \cdots \leq q_m = o_\tau$. Let $w_\tau = w_0 w_1 \dots w_{2m-1}$ be a repetition-free word (i.e., the letters w_i, w_j are pairwise distinct). Define $f_\tau\colon V(\tau) \to 2m+1$ by $f_\tau(i) = 2i$ for every $0 \leq i \leq m$. It is easy to see that w_τ and f_τ satisfy the requirements (with $n = 2m$). □

Example. Consider the term $\tau = (x_1 \,;\, x_2{}^{\smile}) \cdot (y_1 \,;\, y_2)$. Recall that we drew the graph $G(\tau)$ in Figure 1. Let the linear ordering on $V(\tau)$ be $\iota_\tau \leq v \leq u \leq o_\tau$. The map f_τ is illustrated in Figure 2. □

The next lemma is about a connection between $\mathsf{LKL}^{\smile}$ and $\mathsf{RKL}^{\smile}$. Note that $w \in \tau[k]$ means that τ is evaluated in a language algebra.

Lemma 2.5. *For every identity-free ground term τ, there are a word w_τ and a valuation k_τ of the variables of τ such that*

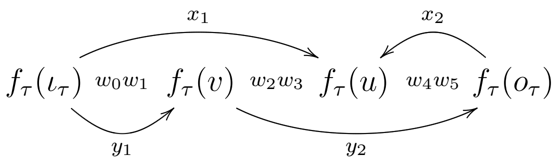

FIGURE 2. The word w_τ and map f_τ for $\tau = (x_1 \,;\, x_2{}^{\smile}) \cdot (y_1 \,;\, y_2)$

(1) $w_\tau \in \tau[k_\tau]$ *and*
(2) *for any term* σ, $w_\tau \in \sigma[k_\tau]$ *implies* $\mathsf{RKL}^{\smile} \models \tau \leq \sigma$.

Proof. Fix τ, and let $w_\tau = w_1 \dots w_n$ and f_τ be as in the statement of Lemma 2.4. For a variable x occurring in τ define

$$k_\tau(x) = \{w_\tau(f_\tau(i), f_\tau(j)) : (i, x, j) \in E(\tau)\}$$

and for a variable x not occurring in τ define $k_\tau(x) = \emptyset$. We show that these w_τ and k_τ are the ones we are looking for.

First, $w_\tau \in \tau[k_\tau]$ by Lemmas 2.3, 2.4 and the construction of k_τ.

For the second statement, assume $w_\tau \in \sigma[k_\tau]$ where σ is a ground term. Then all variables of σ must occur in τ, because otherwise $\sigma[k_\tau] = \emptyset$ by the construction of k_τ. By Lemma 2.3 we have an order-preserving map $h\colon V(\sigma) \to n+1$ such that $h(\iota_\sigma) = 0$, $h(o_\sigma) = n$ and for every edge $(i, x, j) \in E(\sigma)$, we have $w_\tau(h(i), h(j)) \in k_\tau(x)$. Recall that conditions (1)–(3) in Lemma 2.4 are satisfied for w_τ and f_τ. Since w_τ is repetition-free (condition (1)), each subword of length at least 2 occurs in a unique way in w_τ, i.e., for all $i, j, p, q \leq n$ such that $|i - j| \geq 2$:

$$\text{if } w_\tau(i, j) = w_\tau(p, q), \text{ then } i = p \text{ and } j = q. \tag{2.1}$$

Note that the condition $|i-j| \geq 2$ is necessary for (2.1), since $w_\tau(i, i+1) = w_\tau(i+1, i)$ — reading a single letter "backward" is the same as reading it "forward". From (2.1) and the definition of k_τ we infer that the range of h lies inside the range of f_τ as follows. Assume that $(i, x, j) \in E(\sigma)$. Then $w_\tau(h(i), h(j)) \in k_\tau(x)$. Thus $w_\tau(h(i), h(j)) = w_\tau(f_\tau(p), f_\tau(q))$ for some $(p, x, q) \in E(\tau)$ by the definition of k_τ. Since 1 does not occur in τ, then $p \neq q$, and strong injectivity of f_τ (condition (3)) then gives that $|f_\tau(p) - f_\tau(q)| \geq 2$. By (2.1) above then $h(i) = f_\tau(p)$ and $h(j) = f_\tau(q)$. An easy induction shows that for every $i \in V(\sigma)$, there are j and x such that either $(i, x, j) \in E(\sigma)$ or $(j, x, i) \in E(\sigma)$, and we have just seen that in both cases $h(i)$ is in the range of f_τ. Now, by letting $g = h \circ f_\tau^{-1}$ we have that $g\colon V(\sigma) \to V(\tau)$, since f_τ is injective. Finally, we want to show that this g is a homomorphism from $G(\sigma)$ to $G(\tau)$. Let $(i, x, j) \in E(\sigma)$. We have seen that there is $(p, x, q) \in E(\tau)$ such that $h(i) = f_\tau(p)$ and $h(j) = f_\tau(q)$, i.e., $p = g(i)$ and $q = g(j)$, i.e., $(g(i), x, g(j)) \in E(\tau)$. Then indeed $g\colon G(\sigma) \to G(\tau)$ is a homomorphism, since $g(\iota_\sigma) = \iota_\tau$, $g(o_\sigma) = o_\tau$ and g preserves edges labelled by variables. Hence $\tau \leq \sigma$ is valid in relation algebras by [1, Theorem 1].

Finally, assume $w_\tau \in \sigma[k_\tau]$ where σ is not necessarily a ground term. By Lemma 2.1 then $w_\tau \in \delta[k_\tau]$ for some $\delta \in \Gamma(\sigma)$. By the previous case then $\mathsf{RKL}^{\smile} \models \tau \leq \delta$, and by Lemma 2.1 we have $\mathsf{RKL}^{\smile} \models \delta \leq \sigma$. Thus $\mathsf{RKL}^{\smile} \models \tau \leq \sigma$ as desired. □

Discussion of Lemma 2.5. We note that Lemma 2.5 does not remain true if we omit the condition "identity-free" from it. Indeed, let τ be $1 \cdot (x \mathbin{;} y)$. Now if w_τ and k_τ are as in the lemma, then $w_\tau = \lambda$, $\lambda \in k(x)$ and $\lambda \in k(y)$ must hold by $w_\tau \in \tau[k_\tau]$. But then $w_\tau \in \sigma[k_\tau]$ with $\sigma = (1 \cdot x) \mathbin{;} (1 \cdot y)$, though $\mathsf{RKL}^{\smile} \not\models \tau \leq \sigma$. Compare equation (3.1) in Section 3. Similarly, the adjective "ground" is essential in Lemma 2.5. Indeed, take x^+ for τ. By Lemma 2.1, there is $n \in \omega$ such that $w_\tau \in x^n[k_\tau]$. However, $\mathsf{RKL}^{\smile} \not\models x^+ \leq x^n$ for any $n \in \omega$.

We note that if w_τ, k_τ and τ are as in Lemma 2.5, then $\lambda \notin k_\tau(x)$ for any x. Indeed, if $\lambda \in k_\tau(x)$, then $w_\tau \in (1 \cdot x) \mathbin{;} \tau[k_\tau]$, while $\mathsf{RKL}^{\smile} \not\models \tau \leq (1 \cdot x) \mathbin{;} \tau$ for any identity-free ground term τ. Also, we may assume that $k_\tau(x) = \emptyset$ for any variable x not occurring in τ, because if k_τ is not such, then define k'_τ as $k'_\tau(x) = k_\tau(x)$ if x occurs in τ, and $k'_\tau(x) = \emptyset$ otherwise. Then w_τ, k'_τ and τ will satisfy Lemma 2.5.

Finally, if we do not allow converse $^{\smile}$ in the terms, then we can write items (1)–(2) in a more concise form as follows:

$$w_\tau \in \sigma[k_\tau] \quad \text{if and only if} \quad \mathsf{RKL} \models \tau \leq \sigma\,.$$

For $\mathsf{RKL}^{\smile}$ we cannot use this concise form, because the reverse of (2) may not hold. Indeed, let w_τ, k_τ and τ be as in (the proof of) Lemma 2.5, and let σ be $(\tau \mathbin{;} \tau^{\smile})^n \mathbin{;} \tau$ where $n = |w|$. Then $\mathsf{RKL}^{\smile} \models \tau \leq \sigma$. But $w_\tau \notin \sigma[k_\tau]$, since $(\tau \mathbin{;} \tau^{\smile})^n \mathbin{;} \tau[k_\tau]$ only contains words of length at least $4n + 2$. □

3. Equations valid in Kleene lattices

In this section we show that if we add the operation of meet $\cdot$ to the language of Kleene algebras, then the free language Kleene algebra is no more representable as a language algebra, and the equational theories of the language and the relational interpretations differ. We will show exactly how much they differ.

Theorem 3.1. *No free algebra of* LKL *or* RKL *with at least one free generator is representable as a language algebra.*

Proof. In the free algebra, the terms 0, $x \cdot 1$ and 1 are below 1, and all three of 0, $x \cdot 1$ and 1 are different. (For example, $x \cdot 1 \neq 1$ in the free algebra, because if $x = 0$, then $x \cdot 1 = 0 \neq 1$.) However, in a language representation 1 is a one-element set which has only two subsets. □

We note that the free algebra of LKL with no free generator consists of 0 and 1, so is representable as a language algebra. The same holds for the free algebra of RKL with no free generator. It is known that the free Kleene algebra is (isomorphic to) the algebra of regular languages, and the set of regular languages is closed under meet. So the free algebra of Kleene algebras endowed with the operation of meet is no more free in this larger signature. We note that the free LKL-algebras as well as the free RKL-algebras are all representable as relational Kleene algebras. The reason is that a free K-algebra is isomorphic to a subalgebra of a direct product of elements of K, and RKL is closed under subalgebras and direct products (up to isomorphism)[2]. Further, LKL is a subclass of RKL up to isomorphism (we have seen this in the introduction by using the Cayley representation).

Now we turn to equations valid in Kleene lattices. We have seen, by using the Cayley representation, that all the equations valid in relational Kleene lattices (in RKL) are valid in language Kleene lattices (in LKL), too. However, more equations are valid in language Kleene lattices than in relational Kleene lattices. Namely, consider the following equations.

$$(x\,;y)\cdot 1 = (x\cdot 1)\,;(y\cdot 1) \tag{3.1}$$

$$(x\cdot 1)\,;y = y\,;(x\cdot 1) \tag{3.2}$$

$$(z+(x\cdot 1)\,;y)^* = z^* + (x\cdot 1)\,;(z+y)^* \tag{3.3}$$

These equations are all valid in the language interpretations, while they are not valid in the relational interpretations. Equation (3.1) expresses that λ cannot be written as a concatenation of words distinct from λ, while equation (3.2) expresses that $1=\{\lambda\}$ is an atom. It is easy to check that these equations indeed fail in the relational interpretations, see the discussion following the proof of the next theorem.

Theorem 3.2. *Equations* (3.1), (3.2) *and* (3.3) *axiomatize* $\mathsf{Eq}(\mathsf{LKL})$ *over* $\mathsf{Eq}(\mathsf{RKL})$*:*

$$\mathsf{Eq}(\mathsf{RKL}) \cup \{(3.1),(3.2),(3.3)\} \vdash \mathsf{Eq}(\mathsf{LKL})\,.$$

Before proving Theorem 3.2 we prove some lemmas. The following lemma is interesting in itself. It says that if an equation of form $\tau \le \sigma$ distinguishes LKL and RKL, then 1 must occur in τ (perhaps in the form of *). We say that τ is an *identity-free term* if τ is in the language of LKL^-, i.e., 1 and * do not occur in τ, but $^+$ can occur in τ.

Lemma 3.3. *If* $\mathsf{LKL} \models \tau \le \sigma$ *such that* τ *is identity-free, then* $\mathsf{RKL} \models \tau \le \sigma$*.*

[2] Note that the top element is not part of the signature, hence, unlike for Tarski's representable relation algebras, representability on Cartesian squares and on equivalence relations coincide.

Proof. Assume $\mathsf{LKL} \models \tau \leq \sigma$ such that τ is identity-free. Then $\Gamma(\tau)$ consists of identity-free ground terms. Let $\rho \in \Gamma(\tau)$ be arbitrary and let w_ρ and k_ρ be to ρ as in Lemma 2.5, whence $w_\rho \in \rho[k_\rho]$. By Lemma 2.1 we have that $\mathsf{LKL} \models \rho \leq \tau \leq \sigma$. Hence $w_\rho \in \sigma[k_\rho]$. Then $\mathsf{RKL} \models \rho \leq \sigma$ by Lemma 2.5. Finally, since this is true for all $\rho \in \Gamma(\tau)$, by Lemma 2.1 we get that $\mathsf{RKL} \models \tau \leq \sigma$. ☐

We note that Lemma 3.3 is not true with τ and σ interchanged. Indeed, the term $(1 \cdot x) \,;\, y \leq y \,;\, x$ is valid in LKL, but it is not valid in RKL. This implies that 1 does not have to occur on both sides in a "distinguishing" equation: $(1 \cdot x) \,;\, y + y \,;\, x = y \,;\, x$ is a distinguishing equation, and 1 does not occur on both sides.

The next lemma allows to "separate" the use of 1 in terms. We call a term τ to be in *normal form* if τ is of form

$$\eta \,;\, \tau'$$

with either η or τ' possibly missing, such that η is of form $(x_1;\ldots;x_n)\cdot 1$ with $n \in \omega$ and $x_1, ..., x_n$ distinct variables, and τ' is an identity-free term, i.e., 1 does not occur and * occurs only in the form of $^+$ in τ. So this normal form "separates" the use of 1 in a term. We note that the term η behaves like a "switch" in language interpretations: it is the identity 1 if the variables are evaluated to languages which all contain the empty word, and it is zero otherwise. Let $\mathcal{E} = \mathsf{Eq}(\mathsf{RKL}) \cup \{(3.1), (3.2), (3.3)\}$.

Lemma 3.4. *Assume $\mathcal{E}$. Each term is provably equivalent to a finite sum of terms in normal form.*

Proof. By induction on the structure of the terms. We will use the following equation

$$((x \cdot 1) \,;\, y) \cdot z = (x \cdot 1) \,;\, (y \cdot z) \tag{3.4}$$

which is easily seen to be valid in RKL, whence it is in $\mathcal{E}$.

A variable x and 0 are in normal form, and also 1 is in normal form (with $n = 0$). Join is trivial.

Composition: $(\eta \,;\, \tau) \,;\, (\epsilon \,;\, \sigma) = \eta \,;\, \epsilon \,;\, \tau \,;\, \sigma$ by equation (3.2), and $\eta \,;\, \epsilon$ can be written in form $(x_1 \,;\, \ldots \,;\, x_n) \cdot 1$ by equation (3.1).

Meet: $(\eta \,;\, \tau) \cdot (\epsilon \,;\, \sigma) = \eta \,;\, (\tau \cdot (\epsilon \,;\, \sigma)) = \eta \,;\, \epsilon \,;\, (\tau \cdot \sigma)$ by equation (3.4) above and commutativity of meet.

Kleene star: Assume that τ is equivalent to a finite sum of terms in normal form, we want to show that τ^* is also such. Assume $\tau = \sum \eta_i;\tau_i$. We will prove by induction along the number m of i for which η_i is not missing (i.e., it is not the empty term). If $m = 0$, then $\sum \eta_i \,;\, \tau_i$ is identity-free, so $\tau^* = 1 + \tau^+$ where both 1 and $\tau^+ = (\sum \tau_i)^+$ are in normal form (by the definition). Assume now that the statement holds for m, and let $\tau = \sigma + (\epsilon \,;\, \rho)$ where ρ is identity-free and σ is a sum of terms in normal form such that the number of non-missing η_i in σ

is $\leq m$. By the induction hypothesis both σ^* and $(\rho+\sigma)^*$ are finite sums of terms in normal form. Then so is $\epsilon\,;(\rho+\sigma)^*$ by the case for composition. Now, $\tau^* = (\epsilon\,;\rho+\sigma)^* = \sigma^* + \epsilon\,;(\rho+\sigma)^*$ by equation (3.3), and we are done. □

Lemma 3.5. *Assume $\mathcal{E}$. Let $x_1,\ldots,x_n$ be variables, τ be an arbitrary term, and let τ' denote the term we obtain from τ by replacing each occurrence of x_i with x_i+1 for $i \leq n$. Then $\mathcal{E}$ proves*

$$((x_1\,;\ldots;x_n)\cdot 1)\,;\tau = ((x_1\,;\ldots;x_n)\cdot 1)\,;\tau'\,.$$

Proof. By induction on the structure of τ. Let $\eta = (x_1\,;\ldots;x_n)\cdot 1$. The case when τ is the identity constant is trivial.

Variable: We use case distinction according to whether x occurs among $x_1,\ldots,x_n$. If x is distinct from every $x_1,\ldots,x_n$, then the two sides of the equation coincide. Next we assume that $x = x_i$ for some $1 \leq i \leq n$. Clearly $\eta;x_i \leq \eta;x_i+\eta;1 = \eta;(x_i+1)$. For the other direction we have to show that $\eta;1 = \eta \leq \eta;x_i$. Recall that $\eta = (x_1;\ldots;x_i;\ldots;x_n)\cdot 1$. By equation (3.1) we have $\eta = (x_1\cdot 1)\,;\ldots;(x_i\cdot 1)\,;\ldots;(x_n\cdot 1)$ and $\eta;x_i = (x_1\cdot 1);\ldots;(x_i\cdot 1);\ldots;(x_n\cdot 1);x_i$. Since $(x_i\cdot 1);(x_i\cdot 1) = (x_i\cdot 1)$ and the commutativity of composition of sub-identity elements are valid in RKL, we get that $\eta = \eta\,;(x_i\cdot 1) \leq \eta\,;x_i$.

Composition: $\eta\,;\tau_1\,;\tau_2 = \eta\,;\eta\,;\tau_1\,;\tau_2 = \eta\,;\tau_1\,;\eta\,;\tau_2 = \eta\,;\tau_1'\,;\eta\,;\tau_2' = \eta\,;\eta\,;\tau_1'\,;\tau_2' = \eta\,;\tau_1'\,;\tau_2' = \eta\,;(\tau_1\,;\tau_2)'$. We used $\eta \leq 1$ and identity (3.2).

Meet: $\eta;(\tau_1\cdot\tau_2) = (\eta;\tau_1)\cdot\tau_2 = (\eta;\tau_1')\cdot\tau_2 = \eta;(\tau_1'\cdot\tau_2) = \eta;(\tau_2\cdot\tau_1') = (\eta;\tau_2)\cdot\tau_1' = (\eta;\tau_2')\cdot\tau_1' = \eta;(\tau_2'\cdot\tau_1') = \eta;(\tau_1\cdot\tau_2)'$. We used identity (3.4).

Kleene star: First we show that the following equation

$$(\eta\,;\tau)^+ = \eta\,;\tau^+ \tag{3.5}$$

holds. By letting $z=0$, $x\cdot 1 = \eta$ and $y = \tau$ in (3.3) we get

$$(\eta\,;\tau)^* = 1 + \eta\,;\tau^*\,. \tag{3.6}$$

From this we get $(\eta\,;\tau)^+ = (\eta\,;\tau)\,;(\eta\,;\tau)^* = (\eta\,;\tau)\,;(1+\eta\,;\tau^*) = (\eta\,;\tau) + (\eta\,;\tau)\,;(\eta\,;\tau^*) = (\eta\,;\tau) + (\eta\,;\tau\,;\tau^*) = (\eta\,;\tau) + (\eta\,;\tau^+) = \eta\,;\tau^+$. We are ready for our induction step. $\eta\,;(\tau)^* = \eta\,;(1+\tau^+) = \eta + \eta\,;\tau^+ = \eta + (\eta\,;\tau)^+ = \eta + (\eta\,;\tau')^+ = \eta\,;(\tau')^*$.

Join: $\eta;(\tau_1+\tau_2) = \eta;\tau_1+\eta;\tau_2 = \eta;\tau_1'+\eta;\tau_2' = \eta;(\tau_1'+\tau_2') = \eta;(\tau_1+\tau_2)'$ by the additivity of composition. □

We are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Assume that $\mathsf{LKL} \models \tau \leq \sigma$. By Lemma 3.4, $\mathcal{E}$ proves that τ is equivalent to a sum of terms in normal form, say, $\tau = \sum \eta_i\,;\tau_i$. By the equations for join $+$ in $\mathcal{E}$ expressing that $+$ is supremum then it is enough to prove for each i that $\eta_i\,;\tau_i \leq \sigma$. We know that $\mathsf{LKL} \models \eta_i\,;\tau_i \leq \sigma$. Let η_i', τ_i' and σ' be the terms we obtain from η_i, τ_i and σ by replacing all the variables x_j occurring in η_i with x_j+1. Then $\mathsf{LKL} \models \eta_i'\,;\tau_i' \leq \sigma'$, because we get this if we choose any evaluation for the variables occurring in η such that they contain the identity. Since all operations are monotone, we have that $\mathsf{LKL} \models 1 \le \eta_i'$ and $\mathsf{LKL} \models \tau_i \le \tau_i'$. Thus $\mathsf{LKL} \models \tau_i \le \sigma'$. By Lemma 3.3 then $\mathsf{RKL} \models \tau_i \le \sigma'$, since τ_i is identity-free. Hence also $\mathsf{RKL} \models \eta_i;\tau_i \le \eta_i;\sigma'$. So $\mathcal{E}$ proves $\eta_i;\tau_i \le \eta_i;\sigma'$ by $\mathsf{Eq}(\mathsf{RKL}) \subseteq \mathcal{E}$. Also, $\mathcal{E}$ proves $\eta_i;\sigma' = \eta_i;\sigma$ by Lemma 3.5. Now, we get $\eta_i;\tau_i \le \eta_i;\sigma' = \eta_i;\sigma \le \sigma$. The last inequality is by $\mathsf{Eq}(\mathsf{RKL})$. □

Discussion of Theorem 3.2. Equation (3.1) does not follow from $\mathsf{Eq}(\mathsf{RKL}) \cup \{(3.2), (3.3)\}$. Indeed, let U be any set with at least two elements and let A consist of 0, $Id = \{(u,u) : u \in U\}$, $Di = \{(u,v) : u, v \in U, u \ne v\}$ and $U \times U$. Then A is closed under all the operations of RKL (i.e., under $\cdot, +, ;, ^*, 0, 1$). Equation (3.1) is not true in this algebra under the evaluation $x = y = Di$. However, equations (3.2) and (3.3) are true because Id is an atom in this algebra. Indeed, this means that $x \cdot 1$ must be either 0 or 1, and in both cases (3.2) and (3.3) trivially hold.

Equation (3.2) does not follow from $\mathsf{Eq}(\mathsf{RKL}) \cup \{(3.1), (3.3)\}$. Indeed, let $U = \{u, v\}$ be a two-element set, $a = \{(u,u), (u,v)\}, b = \{(u,u)\}$ and $A = \{0, a, b, 1, a+1\}$. Then A is closed under the operations of RKL. Equation (3.2) does not hold in this algebra, take $x = y = a$. Then $(a \cdot 1) \,;\, a = \{(u,u),(u,v)\} \ne \{(u,u)\} = a \,;\, (a \cdot 1)$. It can be checked that (3.1) and (3.3) hold.

We show that equation (3.3) does not follow from equation (3.1) in RKL. Let $U = \{0,1,2\}$ and let $A = \{X \subseteq U \times U : (\forall u, v)[(u,v) \in X \to u \le v\}$. Then A is closed under the operations of RKL, and (3.1) is true in A because one cannot create a new identity pair with composition. However, (3.3) is not true in A as the choice of $x = \{(0,0)\}$, $y = \{(0,1),(1,2)\}$, $z = \emptyset$ shows.

However, (3.3) does follow from (3.2) in RKL, i.e., the equational implication

$$(x \cdot 1) \,;\, y = y \,;\, (x \cdot 1) \to (z + (x \cdot 1) \,;\, y)^* = z^* + (x \cdot 1) \,;\, (z + y)^*$$

holds in RKL, as is easy to see as follows. One can show by using the definition of *, equation (3.2), $(x \cdot 1) \,;\, (x \cdot 1) = x \cdot 1$ and $(x \cdot 1) \,;\, w \le w$ that both $(z + (x \cdot 1) \,;\, y)^*$ and $z^* + (x \cdot 1) \,;\, (z + y)^*$ are equal to the (infinite) sum of products of form $a_0 \,;\, \ldots \,;\, a_n$ where $n \in \omega$, for every $1 \le i \le n$, $a_i \in \{z, y\}$, and $a_0 = (x \cdot 1)$ if at least one of $a_1, \ldots, a_n$ is y and $a_0 = 1$ otherwise.

This motivates the following questions.

Problem 3.6. Is $\mathsf{Eq}(\mathsf{RKL}) \cup \{(3.1), (3.2)\} \vdash (3.3)$ true? Can we use the simpler equation (3.5) introduced in the proof of Lemma 3.5 in place of the more complicated (3.3)?

We also note that Lemmas 3.4 and 3.5 are true for terms containing converse. □

One can check that the proof of Theorem 3.2 is highly modular, so it is true for the star-free reduct of Kleene lattices. A finite set $Ax(\cdot, +, ;, 0, 1)$ is given in [2, Theorem 4.1] axiomatizing the equational theory of the star-free relational interpretations.

Corollary 3.7. *The equational theory of the star-free language interpretations is axiomatized by* $Ax(\cdot, +, ;, 0, 1) \cup \{(3.1), (3.2)\}$.

4. Equations valid in identity-free Kleene lattices

In this section we prove that the equational theories of the identity-free language and relational Kleene lattices coincide and that the free algebra is representable as a language algebra. This shows that we can include meet into the language of Kleene algebras without losing equality of the equational theories if we omit 1 from the language at the same time. So, indeed the differences that we saw in the previous section are caused by the interaction of $\cdot$ with 1.

The following theorem says that if the identity constant is not present, even implicitly in the *-operation, then the same equations hold in language and in relational interpretations.

Theorem 4.1. *The equational theories of* LKL^- *and* RKL^- *coincide:*

$$\mathsf{Eq}(\mathsf{LKL}^-) = \mathsf{Eq}(\mathsf{RKL}^-)\,.$$

Proof. It is enough to see that the same inequalities $\tau \leq \sigma$ hold in RKL^- and LKL^-. Assume $\mathsf{RKL}^- \models \tau \leq \sigma$. Then $\mathsf{LKL}^- \models \tau \leq \sigma$, since we have seen, by using the Cayley representation, that every LKL^--algebra is isomorphic to an RKL^--algebra. On the other hand, if $\mathsf{LKL}^- \not\models \tau \leq \sigma$, then $\mathsf{RKL}^- \not\models \tau \leq \sigma$ by Lemma 3.3. $\square$

One can check that the proof of Theorem 4.1 is again modular, so it works for the star- and identity-free reduct of Kleene lattices. A finite set $Ax(\cdot, +, ;, 0)$ is given in [2, Theorem 4.1] axiomatizing the equational theory of the star- and identity-free relational interpretations.

Corollary 4.2. *The equational theory of the star- and identity-free language interpretations is axiomatized by* $Ax(\cdot, +, ;, 0)$.

Theorem 4.3. *The free algebras of* LKL^- *are representable as language algebras.*

Proof. Let X be any set, $\mathfrak{T}$ be the term algebra generated by X, and let GT^- denote the set of identity-free ground terms with variables from X. For each ground term τ, let w_τ and k_τ be as in Lemma 2.5 on alphabets Σ_τ such that these alphabets are disjoint for distinct terms. Let Σ be the (disjoint) union of these, i.e., $\Sigma = \bigcup\{\Sigma_\tau : \tau \in GT^-\}$. We will represent $\mathfrak{F}$, the free algebra of LKL^- generated freely by X, on the alphabet Σ. We have seen in the discussion after Lemma 2.5 that we may assume that $k_\tau(x) = \emptyset$ for all variables x not occurring in τ. For each $x \in X$ define

$$k(x) = \bigcup\{k_\tau(x) : \tau \in GT^-\}\,.$$

Then $k\colon X \to \wp(\Sigma^*)$. Let k also denote the homomorphism from $\mathfrak{T}$ to $\mathfrak{L}$, the language algebra on Σ, extending this function. We will show that for all terms $\tau, \sigma \in \mathfrak{T}$,

$$k(\tau) = k(\sigma) \quad \text{iff} \quad \mathsf{LKL}^- \models \tau = \sigma\,. \tag{4.1}$$

This will show that the range of k, which is a subalgebra of $\mathfrak{L}$, is a free algebra of LKL^- generated freely by X, hence isomorphic to $\mathfrak{F}$. Now, to prove (4.1), assume that $\mathsf{LKL}^- \models \tau = \sigma$. Then clearly $k(\tau) = k(\sigma)$, since $\mathfrak{L} \in \mathsf{LKL}^-$ and since $k(\delta) = \delta[k]$ for all terms δ. Conversely, assume that $\mathsf{LKL}^- \not\models \tau = \sigma$. Then $\mathsf{LKL}^- \not\models \tau \leq \sigma$ or $\mathsf{LKL}^- \not\models \tau \geq \sigma$, wlog we may assume the former. By Lemma 2.1 then there is a ground term $\delta \in \Gamma(\tau)$ such that $\mathsf{LKL}^- \not\models \delta \leq \sigma$. Now, take the word w_δ. We have that $w_\delta \in \delta[k_\delta]$ by Lemma 2.5. By using disjointness of the alphabets Σ_τ, one can prove by an easy induction that for all ground terms η,

$$k(\eta) \cap \Sigma^*_\eta = \eta[k_\eta]\,. \tag{4.2}$$

Then $w_\delta \in k(\delta) \subseteq k(\tau)$. By $\mathsf{LKL}^- \not\models \delta \leq \sigma$, Theorem 4.1 and Lemma 2.5, we have that $w_\delta \notin \sigma[k_\delta]$. Since the alphabets Σ_δ and Σ_σ are disjoint, we have $w_\delta \notin k(\sigma)$ by (4.2). That is, $k(\delta) \not\leq k(\sigma)$. Hence $k(\tau) \not\subseteq k(\sigma)$, finishing the proof. $\square$

The question arises whether it was necessary to exclude the occurrences of 1 implicit in * in order that we get the positive result Theorem 4.1, or we only needed this in the proof (to ensure that the ground terms in $G(\tau)$ are identity-free). The following equations (4.3), (4.4) show that indeed it was necessary to omit * from the signature of identity-free Kleene lattices.

Consider the following equation:

$$z^* \cdot x \,;\, y = (z^* \cdot x) \,;\, (z^* \cdot y) + (z^+ \cdot x \,;\, y)\,. \tag{4.3}$$

This equation holds in language algebras (i.e., in LKL), since $z^* = z^+ + 1$ and $1 = \{\lambda\}$ is an atom in language algebras. But it does not hold in relation algebras (i.e., in RKL). In fact, the $\leq$-part of equation (4.3) does not hold in RKL as the following example shows. Let us consider the full relation algebra over the base set $U = \{0, 1\}$, i.e., the elements of our algebra are all the binary relations over U. Let $x = z = \{(0, 1)\}$ and $y = \{(1, 0)\}$. Then $x \,;\, y = \{(0, 0)\}$, $z^* = \{(0, 0), (1, 1), (0, 1)\}$ and $z \,;\, z^* = \{(0, 1)\}$. Hence $z^* \cdot x \,;\, y = \{(0, 0)\}$, but $z^* \cdot y = \emptyset$ and $z \,;\, z^* \cdot x \,;\, y = \emptyset$. Thus equation (4.3) does not hold in this algebra. We can see that equation (4.3) is a corollary of (3.1) involving 1, which is valid only in language algebras. Thus indeed the reason for the distinguishing equation (4.3) to work is that the constant 1 implicitly occurs in the operation *. Another equation similar to (4.3) is the following one:

$$(z^* \cdot x) \,;\, y + y \,;\, (z^+ \cdot x) = (z^+ \cdot x) \,;\, y + y \,;\, (z^* \cdot x)\,. \tag{4.4}$$

This is a corollary of equation (3.2).

Let us consider Kleene lattices where we omit explicit use of 1 but we do not omit implicit use of 1, i.e., we do not omit *. Let RKL^* and LKL^* denote the classes of relation and language algebras, respectively, of similarity type $(\cdot, +, ;, ^*)$. We omitted 0 from the similarity type, because $1 = 0^*$ is valid in these algebras, so 0 would bring back 1.

Problem 4.4. Is the equational theory of LKL^* finitely axiomatizable over the equational theory of RKL^*?

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Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, 13–15 Reáltanoda u., 1053 Budapest, Hungary, Tel: 36 1 4838300, Fax: 36 1 4838333

E-mail address: andreka@renyi.hu

URL: http://www.renyi.hu/~andreka/

Department of Computer Science and Information Systems, Birkbeck, University of London, Malet Street, London WC1E 7HX, UK, Tel: 44 20 7631 6738, Fax: 44 20 7631 6727

E-mail address: szabolcs@dcs.bbk.ac.uk

URL: http://www.dcs.bbk.ac.uk/~szabolcs/

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, 13–15 Reáltanoda u., 1053 Budapest, Hungary, Tel: 36 1 4838300, Fax: 36 1 4838333

E-mail address: nemeti@renyi.hu

URL: http://www.renyi.hu/~nemeti/