Lecture Notes on Spectral Analysis of Graphs

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1 Introduction

Recall that given a graph G(V,E), we define its expansion $\alpha(G)$ as,

$$\alpha(G) = \min_{U \subset V} \frac{|\delta(U)|}{\min(|U|, |V-U|)}.$$

Previously, we estimated $\alpha(G)$ via the all-pairs multi-commodity flow problem. Now we will try to estimate $\alpha(G)$ through an eigenvalue analysis.

Definition 1.1 *Given an n by n matrix M, we call λ an eigenvalue of M if there exists $0 \neq x \in R^n$ such that $Mx = \lambda x$. We say x is an eigenvector associated with λ.*

Definition 1.2 *Given an undirected graph G = (V,E) with $|V| = n$, its adjacency matrix A is given by $A_{ij} = 1$ if (i,j) $\in$ E and $A_{ij} = 0$ otherwise.*

Note that the above definition implies that A is symmetric. From now on we will restrict our attention to graphs G that have no self loops, in which case $A_{ii} = 0$.

So what is the combinatorial meaning of Ax? If we think of $x \in R^n$ as an assignment of numbers x_i to each vertex i of G, then Ax corresponds to the operation of assigning to each node in G the sum of its neighbors' values.

Also observe that in the case where G is d-regular A has d ones in each row, so $A\vec{1} = d * \vec{1}$. I.e. if G is d-regular then d is an eigenvalue of A.

Lemma 1.3 *Let G(V,E) be a connected, d-regular graph with adjacency matrix A. If $x \neq c\vec{1}$ is an eigenvector of A with eigenvalue λ, then $\lambda < d$.*

Proof. Let $S = \{i \mid x_i = \max_j x_j\}$. We have $\emptyset \neq S \neq V$ since $x \neq c\vec{1}$. Since G is connected, there exists an edge (i,j) $\in$ E with $i \in S$, $j \notin S$. So $(Ax)_i < d * x_i$ since $(Ax)_i$ is the sum of d components of x, all of which are $\leq x_i$ and at least one of which (namely x_j) that is strictly $< x_i$. So $\lambda < d$. ∎

If G were not connected, then the above lemma is not necessarily true: Let C, V-C be two components of G. let $x = [x_1, \ldots, x_n]$ with $x_i = 1$ if $i \in C$ and zero otherwise. Since C is d-regular, $Ax = dx$. In fact, the number of components of G equals the dimension of the space of eigenvectors with eigenvalue d.

Now we state (without proof) two basic facts about matrices:

Claim 1.4 *Suppose M is a symmetric n by n matrix. Then the eigenvalues of M are real and M has n orthonormal eigenvectors $w_1, \ldots, w_n$ (note that $\{w_i\}$ spans R^n). Furthermore, if λ_i is the eigenvalue associated with w_i, then $M = PDP^T$ where $D = diag(\lambda_1, \ldots, \lambda_n)$ and $P = [w_1, \ldots, w_n]$.*

Claim 1.5 *Let M be an n by n matrix. The following are equivalent:*
(i) If λ is an eigenvalue of M, then $\lambda \geq 0$.
(ii) $x^T M x \geq 0$ for all $x \in R^n$
If these properties hold then we call M positive semi-definite. If M is symmetric then the above are equivalent to,
(iii) $M = N^T N$ for some matrix N.

Lemma 1.6 *Let M be a symmetric, positive semi-definite matrix with orthonormal eigenvectors $w_1, \ldots, w_n$ and associated eigenvalues $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. Let $S = \{x \in R^n \mid x \perp w_1\}$ and let $S_1 = \{x \in S \mid \|x\| = 1\}$. Then,*

$$\lambda_2 = \min_{0 \neq x \in S} \frac{x^T M x}{x^T x} = \min_{x \in S_1} x^T M x.$$

Proof. The second equality follows since we can replace x by $x/\|x\|$.
Take $x \in R^n$. Since $\{w_i\}$ spans R^n, we can find α_i such that $x = \sum_{i=1}^n \alpha_i w_i$. So,

$$Mx = M \sum_{i=1}^n \alpha_i w_i = \sum_{i=1}^n \alpha_i * M w_i = \sum_{i=1}^n \alpha_i \lambda_i.$$

And since $\{w_i\}$ is an orthonormal set,

$$x^T M x = \sum_{i,j} \alpha_i \alpha_j \lambda_i w_i w_j = \sum_{i=1}^n \lambda_i \alpha_i^2.$$

Now, $x \in S_1$ iff $\sum_{i=1}^n \alpha_i{}^2 = 1$ and $\alpha_1 = 0$. And to minimize $\sum_{i=1}^n \lambda_i \alpha_i^2$ subject to $\alpha_1 = 0$ and $\sum_{i=1}^n \alpha_i^2 = 1$ we set $\alpha_2 = 1$, $\alpha_i = 0$ for $i \neq 2$. So,

$$\min_{x \in S_1} x^T M x = \lambda_2.$$

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2 Which matrix do we use?

Up until now, we've been looking at A, the adjacency matrix of G. Another useful matrix is L, the Laplacian matrix of G, where L is defined as follows.

Definition 2.1 *Given a graph $G = (V, E)$ we define the Laplacian matrix L by $L_{ij} = deg(i)$ if $i = j$, $L_{ij} = -1$ if $(i, j) \in E$, and 0 otherwise.*

So if $\phi = diag(deg(1), \ldots deg(n))$, then $L = \phi - A$. Because L has row and column sums equal to 0, 0 is an eigenvalue of L, and $\vec{1}$ is the corresponding eigenvector. This is very useful because it also holds for nonregular graphs, G.

In order to use [1.5], we want to show $L = N^T N$ for some matrix N. We can define N by letting each column correspond to a node $1 \ldots n$, and each row correspond to an edge $e_1 \ldots e_m$, where $|E| = m$, and placing ± 1 in the entries corresponding to edges and endpoints. Specifically, we will set $N_{ij} = 1$ if $e_i = (j, k)$ for some $k > j$, $N_{ij} = -1$ if $e_i = (j, k)$ for some $k < j$, and $N_{ij} = 0$ otherwise.

Thus, $L = N^T N$, so from [1.5] we have $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Notice that here all of our eigenvalues are "flipped", so instead of looking at λ_{n-1}, we are interested in the behaviour of λ_2.

In order to bound λ_2, we will consider $x \in \mathbb{R}^n$, which is a labeling on V. So the new label on node i is $(Lx)_i = \sum_{(i,j) \in E} (x_i - x_j)$. In $x^T Lx$, this means each edge will appear in the sum as $x_i(x_i - x_j) + x_j(x_j - x_i) = (x_i - x_j)^2$, and so $x^T Lx = \sum_{(i,j) \in E} (x_i - x_j)^2$.

Then we can find the first two eigenvalues in the following way:

$$\lambda_1 = \min_{0 \neq x \in \mathbb{R}} \frac{x^T Lx}{x^T x} = \min_{0 \neq x \in \mathbb{R}} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_i x_i^2} = 0,$$

and so for the corresponding eigenvector, ω_1, we have $(\omega_1)_i = (\omega_1)_j \forall i, j$.

$$\lambda_2 = \min_{0 \neq x \in \mathbb{R}, x\omega_1 = 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_i x_i^2}$$

This acts like a 1-D embedding problem where one wants to minimize the strain on the edges. We will use this to obtain upper and lower bounds on λ_2 in terms of $\alpha(G)$. Note that this is equivalent to finding upper and lower bounds on $\alpha(G)$ in terms of λ_2.

Claim 2.2 $2\alpha \geq \lambda_2 \geq \frac{\alpha^2}{2\Delta}$ *where* Δ *=maximum degree of* G.

Notice that for expander graphs, 2α and $\frac{\alpha^2}{2\Delta} = \frac{\alpha}{\Delta}\frac{\alpha}{2}$ are close. But for other graphs, $\frac{\alpha}{\Delta}$ could be quite small. Here we will prove the easier half of this inequality. The other part, while not conceptually difficult, is based on a more elaborate argument using the Cauchy-Schwarz inequality.

Lemma 2.3 $2\alpha \geq \lambda_2$

Proof. Pick any $x \in \mathbb{R}^n$ with $\sum x_i = 0$. Then

$$\lambda_2 \leq \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_i x_i^2}.$$

Look at a partition (A, B), $a = |A| \leq |B| = b$ which achieves α: $\frac{e(A,B)}{a} = \alpha$. Then set $x_i = \frac{1}{a}$ if $x_i \in A$, and $x_i = \frac{-1}{b}$ if $x_i \in B$. So

$$\sum x_i = a\frac{1}{a} + b\frac{-1}{b} = 0.$$

We also have

$$\sum x_i^2 = a\frac{1^2}{a^2} + b\frac{(-1)^2}{b^2} = \frac{1}{a} + \frac{1}{b}.$$

Thus we obtain:

$$\frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_i x_i^2} = \frac{e(A,B)(\frac{1}{a} + \frac{1}{b})^2}{\frac{1}{a} + \frac{1}{b}} = e(A,B)(\frac{1}{a} + \frac{1}{b}) \leq 2\frac{e(A,B)}{a} = 2\alpha.$$

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