1 Single item auction with k identical Bayesian users

This section is a follow-up of the last lecture. We have k users (bidders), user i gets value v_i if he wins the bidding. Every user has this value independent with each other, and all v_i are drawn from the *same* distribution. We have a first-price auction, i.e. the user with highest price will win, and will pay whatever he/she declares.

Theorem 1 *Let*

$$\beta(v) = Exp(\max_{j \neq i} v_j | v_j \leq v, \forall j \neq i) \tag{1}$$

then every user i bidding $\beta(v_i)$ is a Nash equilibrium.

Proof. W.l.o.g., we consider user 1, and his/her valuation v.
Let

$$X = \max_{j>1} v_j \tag{2}$$

and

$$G(v) = Pr(X \leq v) \tag{3}$$

If we want to calculate the conditional expectation $\beta(v)$ as defined in the theorem, we will need $Exp(X)$ and $G(v)$.

To simplify our proof, let us make the assumption that all the functions (probability, $\beta(\cdot)$, $G(\cdot)$, etc.) have whatever smoothness condition (continuous, differentiable, etc.) we required.

It is easy to see that $G(v)$ is monotone increasing, as shown in the following figure.
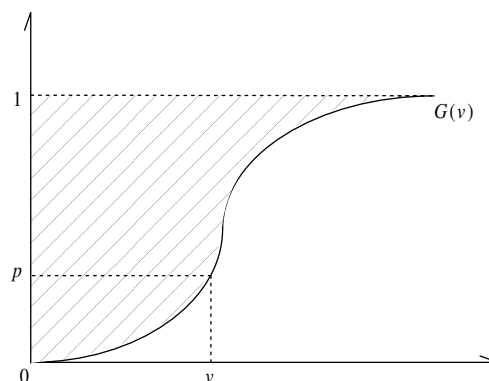
From the figure above, we can make the following two claims:

Claim 2

$$
\begin{aligned}
Exp(X) &= \quad shaded\ area \\
&= \quad \int_0^\infty Pr(X \geq v)dv \\
&= \quad \int_0^1 G^{-1}(p)dp
\end{aligned}
$$

Claim 3

$$
\begin{aligned}
& Exp(X|X \leq v) \cdot Pr(X \leq v) \\
&= \quad \int_0^{G(v)} G^{-1}(x)dx \\
&= \quad v \cdot G(v) - \int_0^v G(x)dx
\end{aligned}
$$

Suppose users $j \geq 2$ bid $\beta(v_j)$, then what should user 1 do? Suppose he/she bids b, then his/her happiness is

$$(v_1 - b) \cdot Pr(b \geq \max_{j \geq 2} b_j) = (v_1 - b) \cdot Pr(b \geq \max_{j \geq 2} \beta(v_j)) \tag{4}$$

The goal is to prove that $b = \beta(v_1)$.

First, we need to show that there exists some z s.t. $b = \beta(z)$. The reason for this is that the maximum useful bid for user 1 is $\leq \max_v \beta(v)$, since it is the maximum possible bid of all the other users, and the $\beta(\cdot)$ function is continous.

Second, we need to show that $z = v_1$. Consider the happiness of user 1, and we will obtain the following result by the monotonicity of $\beta(\cdot)$ and the assumption that all the users have the same distribution:

$$
\begin{aligned}
& (v_1 - \beta(z)) \cdot Pr(\beta(z) \geq \max_{j \geq 2} \beta(v_j)) \\
&= \quad (v_1 - \beta(z)) \cdot Pr(z \geq \max_{j \geq 2} v_j) \\
&= \quad (v_1 - \beta(z)) \cdot G(z) \\
&= \quad v_1 \cdot G(z) - \beta(z) \cdot G(z) \\
&= \quad v_1 \cdot G(z) - z \cdot G(z) + \int_0^z G(x)dx
\end{aligned}
$$

To maximize the happiness of user 1, we take the derivative of the formula above, and let it be zero:

$$
\begin{aligned}
& v_1 \cdot G'(z) - z \cdot G'(z) - G(z) + G(z) = 0 \\
\Rightarrow \quad & v_1 \cdot G'(z) - z \cdot G'(z) = 0 \\
\Rightarrow \quad & v_1 = z
\end{aligned}
$$

■

From above, we know that even if the mechanism is first-price auction, users will pay as if it was a second-price one.

Now we want to generalize this result, i.e. for all the auction mechanisms that satisfy some assumptions, Bayesian users will behave as in a second price auction.

2   Revenue equivalence theorem

First, we need to make several assumptions:

1. In Nash, the person with highest valuation will win. (For identical distributions, this is reasonable).

2. A function $e(p)$, as defined below, is continuous.

   Given all others' strategies, a user will choose a bid b, which affects

   - expected payment, which is not conditioned on winning;
   - probability p of winning.

   Assuming the function from bids b to probabilities p is continuous and invertible, we define $e(p)$ as the expected payment on bid with winning probability p. One of our assumption is that the function $e(p)$ now defined is continuous.

3. The function $e(\cdot)$ satisfies $e(0) = 0$, i.e. if a user does not want to win, he/she will pay nothing.

The user wants to optimize their expected happiness, which is

$$v \cdot p - e(p) \tag{5}$$

where he/she can control probability p. By basic calculus, he/she will choose p s.t. $v = e'(p)$. By assumption 1,

$$p(v) = Pr(v \text{ is the highest}) = G(v) \Rightarrow v = G^{-1}(p) \tag{6}$$

Therefore, we have

$$e'(p) = G^{-1}(p) \tag{7}$$

So

$$e(p) = \int_0^p G^{-1}(x)dx + e(0) = \int_0^p G^{-1}(x)dx \tag{8}$$

So the expected payment of user with value v will be

$$\int_0^{G(v)} G^{-1}(x)dx, \tag{9}$$

which is the same as the bid in the first price auction by Claim 3.

The above result has the hidden restriction that there is always somebody winning the auction, but we can alos allow the auctioneer not to sell when the price is too low.

For example, consider the special case that there is only one bidder, then if the auctioneer has to sell, then his/her income will always be zero, no matter what mechanism is in use. If the auctioneer has the option not to sell to anybody, he/she can declare a price r, and sell the item only if the price is higher than r. The income will be $r \cdot Pr(v \geq r)$. Here, r is called "reserve price". This can be viewed as the auctioneer is also a bidder offering a price r.

There is also a stronger version of the revenue equivalence theorem, which basically says that under the assumptions similar to what was used above, the auctioneer's income only depends on the probability he sets for not selling. For example, this means that there is an auction maximizing the auctioneer's income that is a first price (or second price) auction with some reserve price r.