CS 684 Algorithmic Game Theory	Scribe: Benjamin Ee
Instructor: Eva Tardos	March 15, 2004

We have the following market equilibrium game: There is a link with limited bandwidth ***B*** (set ***B***=1 w/o loss of generality), and *n* users that want a share of the bandwidth. User *i* has utility $U_i(x_i)$ for x_i units of bandwidth, where $U_i(x_i)$ is a continuous, monotonic increasing and strictly concave function.
Objective: Distribute the bandwidth among all the users so that total demand = ***B****.*

Algorithm 1 *(assumes global knowledge of individual utility functions)*

Generate market clearing price **p** for the bandwidth. If utility is also measured in dollars, then:

- User *i* maximises: $U_i(x_i) - x_i\mathbf{p}$. $U_i(x_i)$ being strictly concave results in a unique max x_i.
- Equilibrium: Set **p** s.t. $\sum_i (x_i) = B$. This is the price s.t when each user decides what to do in isolation, we share everything. It is almost a special case of the exchange economy where the $n+1^{th}$ player has 0 utility for everything, and players have no limit in their cash.

Theorem 1 *Let $U_i(x_i)$ be utility function for each of n users demanding bandwidth, total bandwidth be* ***B****, and the unit price of bandwidth be* ***p****. Then exists* ***p*** *such that total bandwidth demanded by users =* ***B****.*

- Let *x* be the total amount of bandwidth demanded.
- Then we seek to maximize $U_{total}(x) - x\boldsymbol{p}$.
- $$\frac{dUtotal}{dx} = U'_{total}(x) - \mathbf{p} = 0.$$
- Since U'_{total} is continuous, there must exist some value of **p** such that the value of x which sets $\frac{dUtotal}{dx}$ to 0 is ***B***.

Theorem 2 *Equilibrium price* ***p*** *results in social optimum ($U_{total}(x) - x\boldsymbol{p}$ is maximized).*

- Consider any allocation $y_1 \ldots y_n$.
- Let allocation resulting from equilibrium price **p** be $x_1 \ldots x_n$.

$$\sum_i (U_i(y_i) - y_i\mathbf{p}) \le \sum_i (U_i(x_i) - x_i\mathbf{p})$$

$$\therefore \sum_i (U_i(y_i)) - \mathbf{p}B \le \sum_i (U_i(x_i)) - \mathbf{p}B.$$

- Disadvantage of this algorithm is that $U_i(x)$ for all *i* is private information, which users are typically either unaware of, or not willing to disclose.

Algorithm 2

- "*Kelly*" pricing mechanism:
    - *Does not assume knowledge of individual utility functions*
    - Start by obtaining total amount each player is willing to pay for the bandwidth.
    - Let amount each player agrees to pay initially be w_i.
    - Set bandwidth that player i received to $\frac{w_i \times B}{\sum_{all j} w_j}$ (proportional fair sharing).
    - Results in implicit unit price of $\mathbf{p} = \frac{\sum_{all j} w_j}{B}$ for bandwidth.
    - Game:
        - Network gives price $\mathbf{p} = \frac{\sum_{all j} w_j}{B}$ to all users.
        - Users use $\mathbf{p}$ to update their w_i. Assume users are price takers, and do not think about their effect on price. This is realistic if users are assumed to be small enough not to affect prices. Users update prices by attempting to maximize $U_i(\frac{w_i}{B}) - w_i$ (analogous to maximizing $U_i(x_i) - x_i\mathbf{p}$ in first algorithm).
        - Users continue updating prices until Nash equilibrium results where no more updates occur.

Theorem 3 *Equilibrium price* $\boldsymbol{p}$ *resulting from this game will equate supply (*$\boldsymbol{B}$*) and total bandwidth demanded by all users at* $\boldsymbol{p}$*.*

- Note that total bandwidth demanded during all iterations of the update loop $= \sum_{all i} (\frac{w_i}{\sum_{all j} w_j}) \times \mathbf{B}$.
  $= \mathbf{B}$.
- Hence, at Nash equilibrium, where users do not want more (or less) bandwidth at the current prevailing prices, then $U_i(\frac{w_i}{B}) - w_i$ is maximized for all users ***at that price***. Therefore, we have the same equilibrium as was reached by the first algorithm.