CS 684: Algorithmic Game Theory — Friday, March 12, 2004
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Network-building

This lecture describes a game that models the building of a network. There are three main points:

1. For the general case of this game, Nash equilibria do not always exist. The finding of Nash equilibria in this game is NP-complete.
2. When Nash equilibria exist in this game, the total cost of building certain Nash-equilibrium networks is between 1 and k (the number of players) times the cost of the optimal network.
3. For a simplified version of this game, the total cost of building the Nash-equilibrium network can be found and is equal to the cost of the optimal network.

For more details, please see E. Anshelevich, A. Dasgupta, É. Tardos, and T. Wexler, "Near-optimal network design with selfish agents", STOC '03, available at http://www.cs.cornell.edu/~wexler/.

Introduction

Previously, Professor Tardos presented Roughgarden games, in which players route traffic in a network. The players' selfish motive: to achieve the shortest routing time.

Today's lecture will take a different perspective: the building of a network.

The game

We start with a graph representing a network $G=(V, E)$, where we can think of the vertices as servers and the edges as *possible* links. Every edge e has some cost $c_e \geq 0$ needed to install the link, connecting the servers at the ends.

We have k players. Each player i has a pair of nodes s_i, t_i, and is interested only in building just enough of the network to connect those nodes, not in building the entire network.

The player strategies are given by the matrix $p_i(e)$, whose elements represent the statements "player i contributes $p_i(e) \geq 0$ towards the cost of edge e."

Bought network

First, we'll check whether edge e is bought. Add up all players' contributions and check whether this sum is at least the cost of the edge. Define the **bought graph** or **bought network** to be the set B of edges e satisfying: $\sum_i p_i(e) \geq c_e$.

To have the players connect their pairs as cheaply as possible, define each player's utility u_i to be:

- $-\sum_{e \in B} p_i(e)$ [minus the total amount paid] if he connects his pair (regardless of whether all the edges were actually built.);
- $-\infty$ [minus infinity] otherwise (thereby forcing players to connect).

Details and Comments

Today, we just want to connect the pair as cheaply as possible. There is no notion of fairness or capacity.

Today,

- The sources and sinks are not necessarily disjoint.
- The graph is undirected (but the directed case is not much different).
- For all edges, set the edge cost = 1.
- There can be nodes that are not terminals. (We will look at Steiner nodes later in the lecture.)

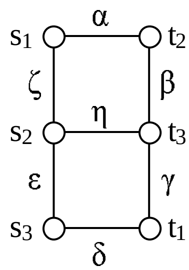
In a real situation, we might have other constraints that we will ignore today, such as:

- Multiple pairs per player
- Need for redundant network links
- Desire for shortest time

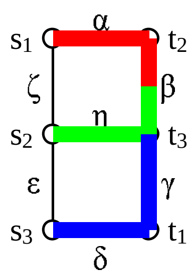
(This will be a full-information game: everyone knows the graph and contributions.)

Plan: Study the Nash Equilibria of this game. What are the stable solutions?

For illustration, consider the following network:

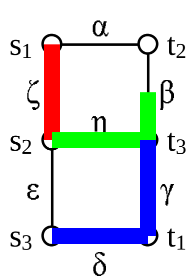


Suppose we arrange the strategies as:



Player	$p_i(\alpha)$	$p_i(\beta)$	$p_i(\gamma)$	$p_i(\delta)$	$p_i(\varepsilon)$	$p_i(\zeta)$	$p_i(\eta)$	u_i
1	1	0.5	0	0	0	0	0	***–1.5***
2	0	0.5	0	0	0	0	1	***–1.5***
3	0	0	1	1	0	0	0	***–2***
Bought?	Yes	Yes	Yes	Yes	No	No	Yes	

What happens? All players have connected their sources and sinks. However, this is not a Nash equilibrium, because player 1 would prefer to switch his strategy to:



Player	$p_i(\alpha)$	$p_i(\beta)$	$p_i(\gamma)$	$p_i(\delta)$	$p_i(\varepsilon)$	$p_i(\zeta)$	$p_i(\eta)$	u_i
1	0	0	0	0	0	1	0	***–1***
2	0	0.5	0	0	0	0	1	***–∞***
3	0	0	1	1	0	0	0	***–2***
Bought?	No	No	Yes	Yes	No	Yes	Yes	

Result:

- player 1's utility goes from –1.5 to –1;
- player 2's utility goes from –1.5 to –∞ (fails to connect).

Burning questions

Do Nash equilibria always exist? If so, how expensive are they (compared to an optimum network)? Can we find these equilibria?

All of these questions have disappointing answers.

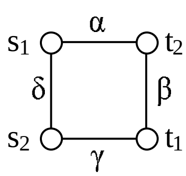
Basic properties of Nash equilibria in this game

Any Nash equilibrium must:

1. buy an acyclic network (a tree or a forest). (If there were a cycle, then players would prefer to buy one fewer edge, not affecting connectivity.)
2. players only contribute to edges on their unique path in the bought network.
3. for any edge e, total payment is either c_e or zero.

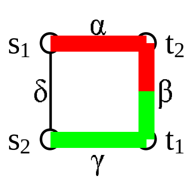
Example with no Nash equilibrium

Consider the following simple example where there is no Nash equilibrium:

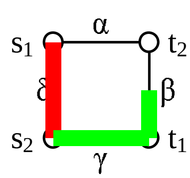


Two players, four nodes (two sinks and two sources), and four edges, each with cost 1. (This is also figure 1 from the paper.)

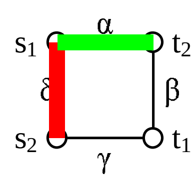
For example, we could begin with both players paying for 1.5 edges:



But then player 1 would prefer to defect:



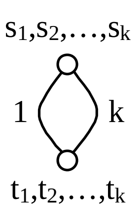
And then player 2 would prefer to change to:



This example shows that Nash equilibria don't necessarily exist.

Example with multiple Nash equilibria

Here's another example: imagine a two-node, two-edge graph, where all k players have the same source and sink nodes. The two parallel links have costs 1 and k, as shown.



There are at least three Nash equilibria:

- Each player could pay 1/k, and the group as a whole buys the cheaper edge.
- Each player could pay 1, and the group as a whole buys the cost-k edge.
- One player could pay 1, buying the cheaper edge, and the other players free-ride.

Calculating the Nash/Opt ratio:

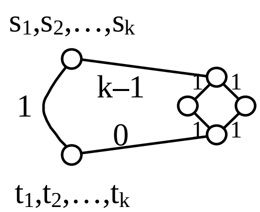
Claim: Any Nash equilibrium costs at most k times the total cost of the optimal solution. (k·cost(OPT)).

Proof: Suppose otherwise — that there exists a Nash equilibrium where the total cost exceeds (k·cost(OPT)). Then there must be at least one player paying more than the optimal total cost. This is a contradiction, because that one player would have preferred to pay just the optimal total cost.

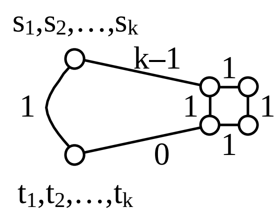
(Recall and compare: in the Roughgarden game, the costs of Nash equilibria were unique, and Nash equilibria always existed.)

Define the **optimistic price of anarchy** to be the ratio of the total cost of the cheapest Nash equilibrium to the total cost of the optimal solution. This quantity indicates how good uncoordinated solutions can be.

Even the best Nash equilibrium can be terrible; for example, combine the two previous examples by inserting the no-equilibrium network into the cost-k edge:



or



(where, as before, the no-equilibrium network players (not included in k) have sources and sinks at opposite corners of the little network) This forces the k players to buy the expensive edge.

Nash claimed that mixed equilibria always exist. In our case, we only consider pure strategies because all players must connect; otherwise, the expected utility is not well-defined. Any given connection is a manifestation of a pure strategy.

Single-source connection game

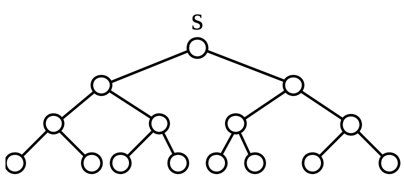
Since finding Nash equilibria is in general NP-complete, we will consider a simpler case:

Define a **single-source connection game** to be one in which all players share the same source node. (For all players *i*, $s_i = s$.) Outside of the game-theoretic context, this is the Steiner tree problem, with the root as the source.

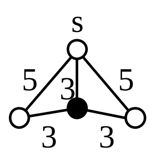
Theorem: in a single-source connection game, Nash equilibria exist, and the optimistic price of anarchy is 1.

Proof sketch:

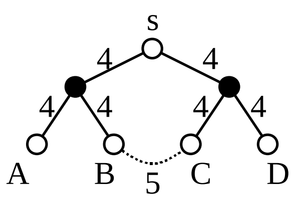
Simple case: all nodes are player-terminals. Imagine starting with a minimum-cost spanning tree. This is stable if every node pays for the edge immediately toward root. If any player were to prefer to buy a different edge (i.e., one not in the minimum-cost spanning tree), then we must not have started with a minimum-cost spanning tree.



Complication: If we add Steiner (non-terminal) nodes (represented in the figures as filled disks), then we must determine a way to pay for the edges from the Steiner nodes toward the root. We must begin with an optimum Steiner tree.



Here, players will pay a maximum of 5.



The payments don't have to be split evenly: players A and D will pay up to 8, while players B and C will only pay up to 5.

The idea is that we can add the payments from the bottom up (from the sinks to the source), while never violating the implicit constraints that a player will pay only as much as her or his cheapest alternative path to the root. This works; below is an argument by contradiction:

What if we ask all the players what they're willing to pay, and it's not enough to buy the optimum Steiner tree? It must be that some player has some cheaper alternate edge to buy than that assigned by the optimum Steiner tree. But if we were to allow this player to deviate, then the total payment for the bought network would be less than for the optimal Steiner tree, which is a contradiction. Either this player or some other player must have lied.

In both cases, the minimum-cost spanning tree or minimum-cost Steiner tree is optimal, and the optimistic price of anarchy is 1.