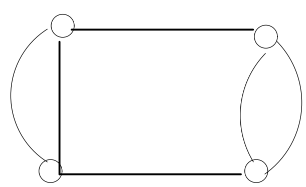
CS 684: Approximation and Network Algorithms	**December 5, 2001**
Problem Set 3	**Due Friday December 14**

(1) In the lecture on November 11 we considered an auction for buying a spanning tree. To recall the setup was this. Edge edge is owned by a separate player. The edge has a true cost only known to the player himself. The goal of the mechanism is to select a minimum cost spanning tree, and pay each edge for its cost plus a premium, where the premium's goal is to induce players to tell their true costs. The mechanism we gave selects the min cost spanning tree T^*, and pays edge $e \in T^*$ using the following formula:

$$cost(MST \text{ in } G - e) - cost(MST) + cost(e).$$

We did not prove this, but this is essentially the only payment scheme that makes selecting the MST truthful. We proved that the total payment to all edges is at most the cost of the MST $\tilde{T}$, of the graph $G - T^*$. This statement is roughly that "if there is strong competition, and there are many small cost spanning trees, than we do not pay much". Here you are asked to prove a stronger version of this statement. Let T_2^* the minimum cost of the union of two spanning trees. Prove that the total payment to all edges the $cost(T_2^*) - cost(T^*)$.

There a few things worth noting here, even though you don't have to worry about them for the problem. First, the new bound proposed here is as least as strong as the old one, as $T^* \cup \tilde{T}$ is the union of two disjoint spanning trees so $cost(T_2^*) \leq cost(T^*) + cost(\tilde{T})$. Further, the min cost union of two disjoint spanning tress T_2^* can be found in polynomial time. Finally, that in same cases the new bound is much stronger than the old one. Here is an example. There are 4 nodes a, b, c, d. The edges drawn on the figure with straight lines cost 1 each, the curved edges cost $1 + \epsilon$, and all edges not drawn cost a lot. Now the straight tree is the MST T^*. one we remove T^* the next tree costs a lot, but there are two small cost disjoint trees, so $cost(T_2^*) = 6 + 3\epsilon$.



(2) On December 5th we discussed the cost sharing problem for trees. Here is a simpler version of the problem. There is a graph $G = (V, E)$ with a root $r \in V$, and we have selected the minimum cost spanning tree T in G. We will use this spanning tree to send messages from r to all other nodes. The question we consider is how to make the nodes $V - r$, that are receiving the messages share the cost of the spanning tree. Note that in lecture we were also concerned about selecting the set of participants in our "multicast", while here we assume all nodes will participate. We are looking for prizes $p(v) \geq 0$ for all $r \neq v \in V$, so that $\sum_v p(v) = cost(MST)$, and that the prizes are "fair". Here is what we will mean by fair for now: for any subset $S \subset V$, the sum of payments assigned to nodes in S may not exceed the min cost MST for the nodes $S + r$. (This notion of fairness is usually referred to by saying that the cost allocation is in the core of the game). Prove that the following simple cost assignments both satisfy the above requirements.

Root the tree at r, and assign to each node $v \neq r$ the cost of the edge from v to its parent in the MST.

(3) Consider the metric traveling salesman problem. You are given n nodes, and distances $d(i, j)$ between them from the previous problem set. Assume that there is a distinguished node r (the home of the salesman), and we want to find a short traveling salesman tour, and make the nodes $v \neq r$ cost share the cost of the selected tour. As before, we want the price $p(v)$ to be nonnegative, and want that to be "fair", that is want that for any subset $S \subset V$ the sum of payments assigned to nodes in S may not exceed the min cost traveling salesman tour for the nodes $S + r$.

It is NP-hard to find the minimum cost tour, and so we cannot expect that this kind of par payments can recover the cost of the tour we found. Give an algorithm that finds a tour, and a fair cost allocation $p(v)$ for $v \in V - r$ that recovers at least 1/2 of the cost of the tour. Hint: use the MST assignments.