CS 684: Approximation and Network Algorithms
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Game theory: Vickrey auctions

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1 Introduction

With this lecture, we begin the game theory section of the course. We will start by analyzing **auctions**, interactions between either one buyer and a group of sellers for a particular good, or one seller of a good with a group of potential buyers. Specifically, we will focus on strategies for players among the group: what price should they bid if they want to buy the good over the others (or maximize their value; these two desires may not be exactly the same), or what price should they offer for selling their good? We will look at three main things in this lecture:

(1) **Vickrey auctions**, or **second price auctions**

(2) A generalization of Vickrey auctions called the **VCG mechanism**, applicable to solving shortest path and MST

(3) **Frugality** issues in auctioning: how much more do we have to pay in order to get the players bidding to be truthful about their costs?

2 Auctions

Suppose we wish to repair the Stewart Avenue bridge, and we want to find a contractor via an auction mechanism. There is a set of n bidders (players) who have an economic cost c_i of doing the job. This may be considered as the minimum amount of money that player i is willing to accept in exchange for repairing the bridge; otherwise they will not profit. Each player i knows its own c_i, but no one else involved does (including ourselves, the buyer). Let p_i be the payment we give to player i.

The utility for player i is

$$u_i = \begin{cases} p_i & \text{if } i \text{ doesn't win (usually equal to 0)} \\ p_i - c_i & \text{if } i \text{ wins} \end{cases}$$

And the goal for player i is to maximize u_i.

The question may arise if this is a reasonable model of the bidders' preferences. For example, if player i really does not like player j but is friends with player k, she may prefer player k winning the bid over player j (even if she can't win), in which case the utility function would be more complicated. We won't analytically consider these matters here.

Along with utility functions, we will restrict the kind of auction under discussion. The auctions we will think about are **sealed-bid**:

(1) Each player submits a bid in a sealed envelope to us.

(2) We open all bids at once, and from this information, decide who wins and how much we pay them.

The simplest kind of sealed-bid auction is the **first price auction**: the lowest bidder wins the auction, and the price we pay is exactly the price they bid. There are a couple of problems with the first price mechanism. For example, there is so little information about the other bidders, it is difficult for a player to know what to bid, except they should at least bid higher than their cost. One possible solution to this uses Nash equilibrium, which will be covered in the next lecture.

We are interested in finding **dominant strategies** for auctions: methods for bidding where players are guaranteed to maximize their utility, no matter what. We will find such a strategy in the second-price (i.e. Vickrey) auction. In a **Vickrey auction**, the lowest bidder wins the auction, but the price paid to the bidder is the amount of the *second lowest* bid. The significant difference here is that a particular player i cannot change her utility by changing her bid; her payment p_i is dependent upon someone else's bidding price.

The following result shows that, in Vickrey auctions, honesty is the best policy. Namely, the players do best when their bidding price is their cost.

Theorem 2.1 *Bidding c_i is the best possible bid for player i, no matter what the other players do.*

Proof. Straightforward case analysis. Given a player i, let T be the lowest bid besides player i's, i.e. $T = \min\{b_j : j = 1, \ldots, n;\ i \neq j\}$.

Case 1. If $c_i < T$: Player i wins the auction if she bids c_i, and the utility gained is $T - c_i$. If player i "lies" by bidding $b_i < c_i$, she either wins or loses. In winning, she still gets just $T - c_i$ utility, and by losing she gets $u_i = 0$. So bidding her cost is just as good as bidding below it.

Case 2. If $c_i > T$: If player i bids c_i, then she doesn't win and $u_i = 0$. If she bids $b_i < c_i$ and wins, then $u_i = T - c_i < 0$. □

We say that the Vickrey auction is "truthful," in that the best bid possible is the actual cost of doing the work. Before we move on, a couple of remarks:

(a) A Vickrey auction chooses the "social optimum"; i.e. the contractor who will use the least cost, the least amount of the world's resources.

(b) We note two interpretations of the payment to player i in a Vickrey auction:

(1) There exists a threshold T (dependent on the bids other than player i) such that i wins if she bids greater than T (and is paid T) and she loses if the bid is less than T (and is paid 0). This interpretation arises from the proof of truthfulness.

(2) The payment to winner i is equal to player i's declared cost + i's marginal value. Note that the bid value b_i is the declared cost, and the marginal value is the cost that would be incurred if i did not participate in the set of bidders, minus the actual cost.

3 The Vickrey-Clarke-Groves mechanism

We will now generalize the auction scheme. Consider a multigraph of two nodes s and t, and each edge (s, t) represents a bid of a different player. The edge has weight equal to the cost for the player (c_i for some i). We may say that a Vickrey auction scheme leads to finding the shortest path from s to t and the MST for this very simple multigraph.

More generally, let us think of an arbitrary G, with edges "owned" by some player. The player incurs cost c_e if the edge e is part of the shortest path/MST.

In general, we will talk about an "optimal structure" as referring to either a shortest path or an MST, as they both can be used in the below.

Assume that no edge is a bridge in G; i.e. no edge is necessary for all MST or shortest path solutions in the graph. The VCG mechanism for an optimal structure is the following:

Vickrey-Clarke-Groves mechanism:

(1) Players submit sealed bids b_e for each edge $e \in E$.

(2) We select an optimal structure according to these reported costs using the second lowest bids.

(3) Of those edges $e \in E$ that we decide to use, we set

$$p_e = [\text{cost of structure in } G - \{e\}] - [\text{cost of structure}] + b_e.$$

We can also think of p_e as being the cost given that e is deleted from G minus the cost assuming $b_e = 0$.

End Vickrey-Clarke-Groves.

Theorem 3.1 *The VCG mechanism is truthful.*

Proof. Basically the same kind of proof as in the Vickrey auction truthfulness: a threshold argument is used for each edge of G. □

4 Problems with the VCG mechanism

Consider a multigraph with just two nodes s and t again, with two edges (s, t): one edge has weight 1 and the other has weight 100. In a Vickrey auction, one player wins with bid at 1, but is paid 100. We need some form of "competition" to not overpay the winning bidder. But on the other hand, if players know something about the bids, they will try to cheat anyway.

In another example, G has a path of $n+1$ nodes and n edges from s to t, each of bid 1, and a single edge (s, t) with bid $(1+\epsilon)n$. The VCG mechanism will choose the shortest path of n edges, and for each edge e in this path, $p_e = (1+\epsilon)n - n + b_e = \epsilon n + b_e = 1 + \epsilon n$, assuming the players bid truthfully. So the total payment is $(1+\epsilon n) \cdot n$; we overpay by a $(1+\epsilon n)$ factor.

There are other kinds of truthful mechanisms we could use for finding a shortest path or MST; are any of them capable of fixing this problem of arbitrarily large overpaying? The answer is no, and it will be (well... was) discussed in the Theory Seminar on 12/3. (See A. Archer and E. Tardos' work on *Frugal Path Mechanisms* in SODA 2002.)

The good news of the matter is that the problem of finding MSTs using this mechanism doesn't suffer from the overpaying problem that shortest paths do.

Theorem 4.1 *(Bikchandani, Vohra, Schummer, de Vries; 2001) Suppose T^* is the MST of G and $G \setminus T^*$ is connected. Let $\tilde{T}$ be the MST of $G \setminus T^*$. Then the VCG mechanism pays at most the cost of $\tilde{T}$.*

First we present a lemma. Let T^* be the MST of G, $\tilde{T}$ the MST of $G \setminus T^*$.

Lemma 4.2 *Set $p \geq 0$. Suppose k edges of T^* are paid at least p. Then there exist k edges of $\tilde{T}$ whose cost is at least p.*

Proof (of Lemma). Consider the k edges given $\{e_1, \ldots, e_k\}$. When they are removed from T^*, there are $k+1$ components left over.

We claim that any edge of G (other than the e_i) with endpoints in separate components of $T^* \setminus \{e_1, \ldots, e_k\}$ has cost at least p. Suppose otherwise, and that there is such an edge e' with $c_{e'} < p$. Consider $T^* \cup \{e'\}$, which contains a cycle. This cycle must have at least one e_j, for some j (since the endpoints of e' are in separate components of $T^* \setminus \{e_1, \ldots, e_k\}$). Let $T' = T^* \cup \{e'\} - \{e_j\}$, which is also a spanning tree. Note that $cost(T') = cost(T^*) + c_{e'} - c_{e_j}$. Now we claim that e_j was paid less than p. Recall that the payment to e_j is:

$$[\text{cost of MST in } G - \{e_j\}] - [\text{cost of real MST}] + c_{e_j}.$$

Note that $[\text{cost of MST in } G - \{e_j\}] \leq cost(T')$, and

$$cost(T') - cost(T^*) + c_{e'} = (c_{e'} - c_{e_j}) + c_{e_j} = c_{e'}.$$

Therefore the payment to e_j is at most $c_{e'} < p$, which contradicts the assumption of the lemma.

Finally, because $\tilde{T}$ is a spanning tree, there are k edges with endpoints in separate components of $T^* \setminus \{e_1, \ldots, e_k\}$, so they have cost at least p. □

Proof (of Theorem). Label the edges of T^* to be $\{e_1, \ldots, e_{n-1}\}$, such that $p_{e_1} \geq p_{e_2} \geq \cdots \geq p_{e_{n-1}}$ (so, in decreasing order of payment). Label edges of $\tilde{T}$ as $\{f_1, \ldots, f_{n-1}\}$, such that $c_{f_1} \geq c_{f_2} \geq \cdots \geq c_{e_{n-1}}$ (so, in decreasing order of cost). By the lemma, $c_{f_i} \geq p_{e_i}$ for all $i = 1, \ldots, n-1$, so the cost of $\tilde{T}$ is at least the total payment. □

Here is a stronger version of the theorem which is still true:

Theorem 4.3 *Let D be a set of edges consisting of two edge-disjoint spanning trees. Then the total payment is at most the cost of D minus the cost of T^*, the MST.*

We won't prove this here. The above theorems also work for matroids as well; the arguments are similar.