CS 684: Approximation and Network Algorithms	**11/16/01**
Connectivity Network Design #2	**scribe: Tom Wexler**

1 The Connectivity Problem

We are given a graph $G = (V, E)$ with costs $c_e \geq 0$ for all edges $e \in E$ and with connectivity requirements $r_{ij} = r_{ji}$ for all pairs of nodes $i, j \in V$. We want to find a subset of edges $F \subseteq E$ such that for all $S \subseteq V$, $i \in S$, and $j \notin S$, we have that $\delta_F(S) \geq r_{ij}$ and $\sum_{e \in F} c_e$ is minimal over all F satisfying this condition.

The linear relaxation of the corresponding integer problem is as follows:
minimize

$$\sum_e c_e x_e$$

subject to

$$\begin{aligned} 0 \leq x_e \leq 1 & \qquad \forall e \in E \\ \sum_{e \in \delta(S)} x_e \geq f(S) = \max_{i \in S, j \notin S} r_{ij} & \qquad \forall S \subseteq V \end{aligned}$$

Claim 1.1 *This LP can be solved in polytime.*

Proof: The primary concern here is that this LP has an exponential number of constraints. The ellipsoid method can handle such a situation if we have an efficient algorithm which, when given a potential solution x, either reports that x is feasible or finds an inequality that is violated or tight. Such a check can be preformed using min cuts as follows. For every pair of vertices i and j, find the min ij-cut in G with the capacity of edge e defined to be x_e. Now simply compare the value of this cut with the sum of the demands r_{kl} crossing the cut. This effectively determines whether or not there exists a set S such that $\sum_{e \in \delta(S)} x_e \leq f(S)$. $\square$

Claim 1.2 *We can find a vertex solution to for this LP.*

The algorithm we would like to use would be the following.

Algorithm 1.3 *Generate a vertex solution x to the linear program above. For every edge e, we will have that $0 \leq x_e \leq 1$. We choose to take (or buy) those edges e for which $x_e \geq 1/3$. Call this set of edges E_0. Then recursively write the inequalities for $e \in E \setminus E_0$*

$$\begin{aligned} 0 \leq x_e \leq 1 & \qquad \forall e \in E \setminus E_0 \\ \sum_{e \in \delta(S)} x_e \geq f(S) - \delta_{E_0}(S) & \qquad \forall S \subseteq V \end{aligned}$$

and repeat; solve the LP, select all edges assigned a value of at least $1/3$*, and recurse on the remaining edges until all connectivity requirements are satisfied. We call the new set of inequalities generated at each stage the residual LP.*

We need to show that this algorithm works, (i.e. if at any point the connectivity requirements have not yet been met, then any vertex solution has some edge e with $x_e \geq 1/3$) and that the algorithm is a 3-approximation. We prove the latter statement now.

Lemma 1.4 *If the algorithm finds a solution at all, such a solution is a* 3*-approximation.*

Proof: We proceed by induction on the number of edges. Let x be a vertex solution and $E_0 = \{e \mid x_e \geq 1/3\}$. We define $\overline{x}$ to be x restricted to $E \setminus E_0$. Note that $\overline{x}$ satisfies the residual LP. Let F be the set of edges selected by our algorithm after E_0 has been selected. Our solution is $E_0 \cup F$. By our (unstated) induction hypothesis, the cost of F,

$$\sum_{e \in F} c_e \leq 3 \sum_{e \in E \setminus E_0} \overline{x}_e c_e.$$

Hence the total cost of our solution

$$\sum_{e \in E_0} c_e + \sum_{e \in F} c_e \leq \sum_{e \in E_0} c_e + 3 \sum_{e \in E \setminus E_0} \overline{x}_e c_e \leq 3 \sum_{e \in E_0} x_e c_e + 3 \sum_{e \in E \setminus E_0} x_e c_e.$$

This implies that our solution has a cost of at most 3 times the cost of an optimum LP solution LP_{OPT}. Since $LP_{OPT} \leq OPT$, our solution has a cost at most 3 times the cost of an optimum integer solution. □

The remainder of this lecture will be concerned with proving that if x is a vertex solution to the LP, then there exists an edge e such that $x_e \geq 1/3$.

Lemma 1.5 *For any* $E_0 \subseteq E$*, and* $A, B \subseteq V$

$$\sum_{e \in \delta(A)} x_e + \sum_{e \in \delta(B)} x_e \geq \sum_{e \in \delta(A \cap B)} x_e + \sum_{e \in \delta(A \cup B)} x_e,$$

and

$$\sum_{e \in \delta(A)} x_e + \sum_{e \in \delta(B)} x_e \geq \sum_{e \in \delta(A \setminus B)} x_e + \sum_{e \in \delta(B \setminus A)} x_e.$$

Proof: Proof by inspection. We will consider every edge and count how much that edge contributes to the two sides of the inequality. There are only 4 (using symmetry) types of edges that need to be examined, as shown in Figure 1. For example, consider an edge with one endpoint in $A \setminus B$ and one endpoint in $B \setminus A$. Such an edge contributes 2 to

$$\sum_{e \in \delta(A)} x_e + \sum_{e \in \delta(B)} x_e,$$

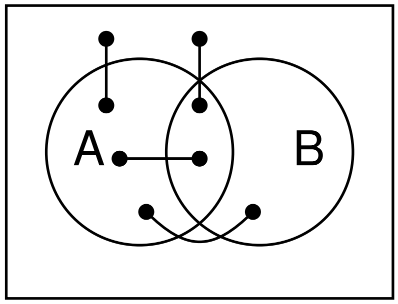

Figure 1: Edge types for consideration in proofs concerning modularity.

contributes 0 to

$$\sum_{e\in\delta(A\cap B)} x_e + \sum_{e\in\delta(A\cup B)} x_e,$$

and contributes 2 to

$$\sum_{e\in\delta(A\setminus B)} x_e + \sum_{e\in\delta(B\setminus A)} x_e.$$

We can similarly bound the amount that each type of edge contributes to the three sums. The lemma follows. □

Definition 1.6 *A function g on sets is* weakly supermodular *if for all $A, B \subseteq V$, either*

$$g(A) + g(B) \le g(A \cup B) + g(A \cap B),$$

or

$$g(A) + g(B) \le g(A \setminus B) + g(B \setminus A).$$

Lemma 1.7 *f is weakly supermodular.*

Proof: Again this proof is by inspection. Without loss of generality assume $f(A) \ge f(B)$. Suppose $f(A) = r_{ij}$, $i \in A$, and $f(B) = r_{kl}$. We know that either $i, j \in B$, $i, j \notin B$, or $i \in B, j \notin B$ ($r_{ij} = r_{ji}$ so we don't need 4 cases). Likewise we have three cases for k and l. Suppose $i, j \in B$, $l \in A$, and $k \notin A$. Then $f(B \setminus A) \ge r_{ij}$ since $j \in B \setminus A$ but $i \notin B \setminus A$. Furthermore, $f(A \setminus B) \ge r_{kl}$ since l is in $A \setminus B$ but k is not. These results together imply that in this case, $g(A) + g(B) \le g(A \setminus B) + g(B \setminus A)$. The remaining cases follow similarly. □

Corollary 1.8 *For any set E_0, $f(S) - \delta_{E_0}(S)$ is also weakly supermodular.*

From this point on, the only characteristic of f that we will rely upon is it's supermodularity. We now want to show that for any weakly supermodular function f, a vertex solution satisfying our constraints always has an edge e assigned $x_e \ge 1/3$. To prove this we first nee a few lemmas.

Without loss of generality we will assume that every edge is fractional, as any edge with $x_e = 0$ can be removed from the graph, and if there is an edge with $x_e = 1$ then clearly the claim holds. Since x is a vertex, there must be $|E|$ sets S such that

$$\sum_{e \in \delta(S)} x_e = f(S)$$

is a tight equation and these inequalities are linearly independent.

Lemma 1.9 *If A and B are such tight sets then either both $A \cap B$ and $A \cup B$ are tight or both $A \setminus B$ and $B \setminus A$ are tight.*

Proof: Suppose $f(A) + f(B) \leq f(A \setminus B) + f(B \setminus A)$.

$$\begin{aligned} f(A) + f(B) &\leq f(A \setminus B) + f(B \setminus A) \\ &\leq \sum_{e \in \delta(A \setminus B)} x_e + \sum_{e \in \delta(B \setminus A)} x_e \\ &\leq \sum_{e \in \delta(A)} x_e + \sum_{e \in \delta(B)} x_e \\ &\leq f(A) + f(B) \end{aligned}$$

Therefore $A \setminus B$ and $B \setminus A$ are both tight, since $f(A \setminus B) + f(B \setminus A) = \sum_{e \in \delta(A \setminus B)} x_e + \sum_{e \in \delta(B \setminus A)} x_e$ and neither left hand term can exceed the corresponding right hand term. Likewise, it is easy to show that if $f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$ then $A \cap B$ and $A \cup B$) are both tight. $\square$

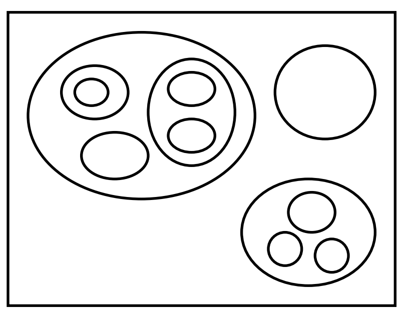

Figure 2: A laminar set system.

Definition 1.10 *A set system $\mathcal{L}$ is laminar if for any sets $A, B \in \mathcal{L}$, either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$ (see Figure 2).*

Lemma 1.11 *There exists a laminar set system $\mathcal{L}$ such that $|\mathcal{L}| = |E|$, for any set $S \in \mathcal{L}$*

$$\sum_{e \in \delta(S)} x_e = f(S),$$

and these equations are linearly independent.

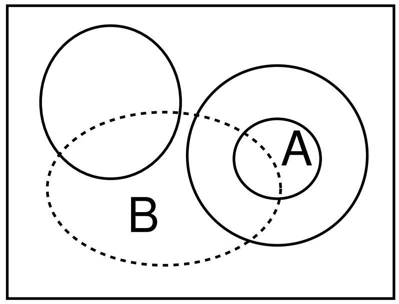

Figure 3: A violation of laminarity in the set system construction.

Proof: We will add sets to $\mathcal{L}$ one at a time, maintaining laminarity. To begin with, take any tight set. Now we have some set $\mathcal{L}$, but $|\mathcal{L}| < |E|$. Since x is a vertex, we know that there exists a tight B that is linearly independent from $\mathcal{L}$. However, it is possible that $\mathcal{L} \cup \{B\}$ is not laminar, as shown in Figure 3. Let A be the smallest set which violates laminarity with B. We know $A \cap B \neq \emptyset$, $A \not\subseteq B$, and $B \not\subseteq A$. We also know that either $A \cap B$ and $A \cup B$ or $A \setminus B$ and $B \setminus A$ are tight. Observe that all of these four sets violate laminarity in strictly fewer places than B does. Therefore, we need to simply show that among the pair $A \cap B$ and $A \cup B$ there is a set linearly independent from $\mathcal{L}$, and likewise among $A \setminus B$ and $B \setminus A$. This would imply that there is a set which has fewer laminarity conflicts than B, is linearly independent from our laminar system, and is tight. Recursively applying this procedure will eventually generate a set with no laminarity conflicts, while still maintaining linear independence and tightness. As before, we will only prove one of the cases.

Suppose $f(A) + f(B) \leq f(A \setminus B) + f(B \setminus A)$. We know that both A and B are tight sets. Therefore, as done in Lemma 1.9, we can derive that

$$\sum_{e \in \delta(A \setminus B)} x_e + \sum_{e \in \delta(B \setminus A)} x_e = \sum_{e \in \delta(A)} x_e + \sum_{e \in \delta(B)} x_e.$$

Note that all edge types are counted equally often on both sides of the equation, except for those edges from $A \cap B$ to $\overline{A \cup B}$, which only contribute to the right hand side of the equation. Thus, all such edges e must have $x_e = 0$ for the equality to hold. But if this is the case, then (by considering the various edge types), we can express the inequality for B

$$\sum_{e \in \delta(B)} x_e \geq f(B) \tag{1}$$

in terms of the following three inequalities

$$\sum_{e \in \delta(A)} x_e \geq f(A) \tag{2}$$

$$\sum_{e \in \delta(A \setminus B)} x_e \geq f(A \setminus B) \tag{3}$$

$$\sum_{e \in \delta(B \setminus A)} x_e \geq f(B \setminus A) \tag{4}$$

In particular (1) = (3) + (4) − (2).

Suppose both inequalities (3) and (4) are dependent. Then (1), the inequality for B, is dependent as well, contradicting our initial assumption. Therefore we can conclude that at least one of the inequalities corresponding to $A \setminus B$ and $B \setminus A$ is independent. Likewise, if we assume $f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$ then we can show that at least one of the inequalities corresponding to $A \cap B$ and $A \cup B$ is independent. □

We now present the main theorem we have been aiming for.

Theorem 1.12 *For any weakly supermodular f, any vertex solution x satisfying*

$$0 \leq x \leq 1$$

$$\sum_{\delta(S)} x_e \geq f(S)$$

has an edge e with $x_e \geq 1/3$. (We can also prove $x_e \geq 1/2$ although we will not do so in class).

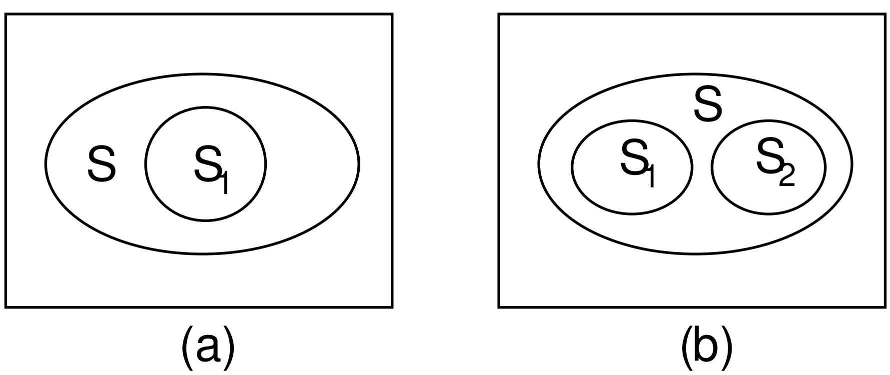

Figure 4: Possible structures of subsets within S.

Proof: Consider a laminar set system $\mathcal{L}$ as described in the previous lemma. We will derive a contradiction by assuming that all edges in a fractional solution can be strictly less that

1/3. Note that if we have a fraction solution using edge values $< 1/3$, then every proper subset of nodes must have at least 4 edges leaving it.

Define $\mathcal{L}(S)$ to be the collection of sets in $\mathcal{L}$ which are subsets of S. We then want to show that the number of edge ends in S is at least $2|\mathcal{L}(S)| + 2$. We do so by induction.

The base case says that for any minimal set $S \in \mathcal{L}$, the number of edge ends in S is at least $2|\mathcal{L}(S)| + 2 = 2 + 2 = 4$. We have already noted that this is true for any set S.

Now consider a general set S. Either S has multiple maximal children, as in Figure 4(a), or it has only one, as in Figure 4(b). Note that it must have at least one, since otherwise S must have been dealt with in the base case.

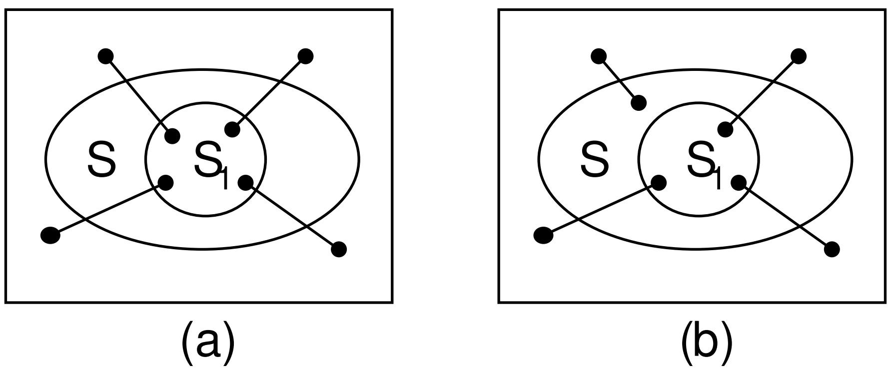

Figure 5: Counting the number of edges ending in $S \setminus S_1$.

Case 1: S has $k \geq 2$ maximal children. Then $\mathcal{L}(S) = S \cup \mathcal{L}(S_1) \cup \mathcal{L}(S_2) \cup ... \cup \mathcal{L}(S_k)$. The number of edge ends is at least the number of edge ends within the disjoint children of S, which by induction is at least $2|\mathcal{L}(S_1)| + 2|\mathcal{L}(S_2)| + ... + 2|\mathcal{L}(S_k)| + 2k = 2(|\mathcal{L}(S)| - 1) + 2k \geq 2|\mathcal{L}(S)| + 2$.

Case 2: S has 1 maximal child, S_1. Then we want to show that there are at least 2 edges ending in $S \setminus S_1$. Suppose not. Then there are either 0 or 1 such edges. If there are no such edges, as shown in Figure 5(a), then the set of edges which leaves S is the same as the set which leaves S_1. But then the inequalities corresponding to S and S_1 are the same, which contradicts our assumption of independence. If there is a single edge ending in $S \setminus S_1$ (Figure 5(b)), then $|f(S) - f(S_1)| = x_e$. However, we know the lefthand side to be an integer by definition, and the righthand side is a fraction by assumption. This is a contradiction.

Therefore if there are no edges with weight of at least 1/3, then there must be at least $2|\mathcal{L}(S)| + 2$ edge ends. But there are exactly $2|E| = 2|\mathcal{L}(S)|$ edge ends. This is clearly a contradiction, and therefore there must exist some edge with weight at least 1/3. □

Thus our algorithm always finds an edge with weight at least 1/3, and therefore, by our earlier result, is a 3-approximation.