CS 684: Approximation and Network Algorithms
October 10, 2001
Brief introduction to linear programming and duality

Frans Schalekamp

Taking the transshipment flows with ∞-capacity as an example, we want to generalize what we did there.

General form

$$\boxed{\begin{array}{ll} \text{Find } x & Ax = b \\ & x \geq 0 \\ & \min cx \end{array}}$$

Analog of transshipment flows (with ∞-capacity):

$$\begin{array}{ll} \text{Find } f_e & f^{in}(v) - f^{out}(v) = b_v \ \forall v \\ & f_e \geq 0 \\ & \min \sum_e f_e c_e \end{array}$$

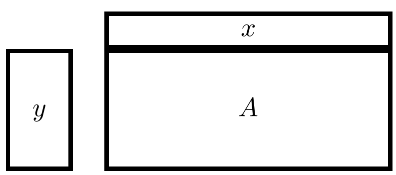
Our flows had

- prices $p(v)$
- reduced costs $c_e^p = c_e + p(v) - p(w)$ (with $e = (v, w)$)

And we had the following lemma:

Lemma 1 *f min-cost flow with cost c, iff min cost with c^p.*

These generalize in the following way:
Let A be $(n \times m)$ matrix (so x is a n-vector).



Now for every m-vector y we can define $c^y = c - yA$.

Lemma 2 *c^y and c-cost are equivalent.*

Proof: For x feasible

$$c^y x = (c - yA)x = cx - yAx = cx - yb.$$

Now note that yb does not depend on x. So minimizing cx or $c^y x$ is equivalent. $\square$

To see this is indeed the analog of lemma 1, remember that in the flow matrix the columns represent the edges, with an 1-entry for the out-node and a -1-entry for the in-node (i.e. if $e = (v, w) \in E$ then in the column of A representing e, there is a -1 on the v's coordinate and an 1 on the w's coordinate).

With flows, our next step was, if $c^y \geq 0 \ \Rightarrow \ \min c^y x \geq 0$.

Corollary 3 *If y such that $c^y \geq 0$ then $\min cx \geq yb$.*

So we have found a *lower bound* on cx. Now we define our **dual LP** to be finding the *best* lower bound, i.e.

$$\begin{array}{ll} \text{Find } y & c^y \geq 0 \ (\Rightarrow \ c \geq yA) \\ & \max yb \end{array}$$

Theorem 4

$$\begin{array}{ccccc} \min & cx & = & \max & yb \\ x \geq 0 & & & yA \leq c & \\ Ax = b & & & & \end{array}$$

Proof: (Weak duality) min $\geq$ max. See Corollary 3.
(Strong duality) min = max. We don't proof this here for the general setting. (For special cases, this is often easily proved.) $\square$

Aside: other form of LP

$$\min cx$$

$$\text{s.t.} Ax \geq b$$

If $y \geq 0 \Rightarrow yA = c$.

Example

$$\begin{array}{cccccc} & & & & & y \\ x & + & 2y & \geq & 7 & \frac{1}{3} \\ 2x & + & y & \geq & 8 & \frac{1}{3} \\ \min x + y & & & & & \end{array}$$

Remark: Obviously we have $y \geq 0$, otherwise the sign changes. With this y we have found that $\min x + y \geq \frac{1}{3} \times 7 + \frac{1}{3} \times 8 = 5$, i.e. $\min cx \geq yb$.
Dual LP *best* such lower bound:

$$\begin{array}{ll} \text{Find } y \geq 0 & yA = c \\ & \max yb \end{array}$$

Fact 5 *Dual of dual LP = primal LP.*

Fact 6 *In the special case of min-cost flows, we have proved strong duality.*

To see this, note that in the special case, the prices $p(v)$ for $v \in V$ are the dual variables. If for all "original" edges (i.e. the edges in the original graph, not the residual graph), $c_e^p \geq 0$, then $\min cf \geq \sum_{v \in V} p(v) b_v$.
We know $\min cf \geq \max\{\sum_{v \in V} p(v) b_v : c_e^p = 0\}$.

Fact 7 *Let p and f be such that $cf = \sum_v p(v) b_v$. Then*

- *f min cost*
- *p optimal solution to dual LP*
- *strong duality true*

In the general setting we have

$$\begin{array}{ccccc} \min & cx & = & \max & yb \\ x \geq 0 & & & yA \leq c & \\ Ax = b & & & & \end{array}$$

Fact 8 *Let x and y be feasible and $cx = yb$. Then*

- *x min cost*
- *y max value*
- *strong duality true*

Look at the weak duality proof again:

$$cx \geq (yA)x = y(Ax) = yb$$

where the first inequality is because $yA \leq c$.
So $cx = yb$ iff $cx = (yA)x$. These are exactly the *complementary slackness* conditions:

$$\boxed{\begin{array}{c} x \text{ and } y \text{ are optimal solutions, iff} \\ x_i > 0 \Rightarrow yA_i = c_i \\ \text{where } A_i \text{ is the } i\text{-th column of } A. \end{array}}$$

What is complementary slackness for min cost flows (∞-capacities)? No capacities $\Rightarrow$ prices are feasible: $c_e^p \geq 0$ all original edges. *Optimal* if $f_e > 0 \Rightarrow c_e^p = 0$.
A quick reminder what we did: we said f and p are at *equilibrium* if

- $f_e < u_e \Rightarrow c_e^p \geq 0$ (where u_e is the capacity, so in the uncapacitated case this is always true)
- $f_e > 0 \Rightarrow c_e^p \leq 0$.

Fact 9 *What we called equilibrium is equivalent to complementary slackness conditions.*

Corollary 10 *Min-cost flow algorithms find prices that solve price LP.*

Suppose we want to solve

$$\begin{aligned} Ax &= b \\ x &\geq 0 \\ \min cx&. \end{aligned}$$

One way to solve this: *primal-dual algorithm.*
Dual

$$\begin{aligned} yA &\leq c \\ \max yb&. \end{aligned}$$

Algorithm

- find feasible x and y so that x & y satisfy complementary slackness, i.e. $x_i > 0 \Rightarrow yA_i = c_i$.
- Start with feasible y (i.e. any y such that $yA \leq c$).
- Try to solve

  $$\bar{A}\bar{x} = b\bar{x} \geq 0$$

  where $\bar{A}$ are those columns of A that have equality in $yA \leq c$, and $\bar{x}$ the corresponding subvector of x.

  In special case of flows $\bar{x}$ are flow variables on edges with $c_e^p = 0$.

  If such $\bar{x}$ exists, *done*. Else improve y.