CS 683 Spring 2001 – Problem Set 3

Due in class Thursday, March 29

(1) Having shown that

$$\frac{\eta(G, \{(s_i, t_i)\})}{\lambda^*(G, \{(s_i, t_i)\})} = O(\log n),$$

for all instances of multicommodity flow in undirected graphs on n nodes, it's natural to ask whether an analogous statement can be made for directed graphs. This remains a very interesting open question — but in order to ask the question properly, we have to make sure we come up with reasonable analogues of η and λ^* in the directed case.

Here are two attempts to do this; we assume G is a directed graph on n nodes, with unit edge capacities, and that $(s_1, t_1), \ldots, (s_k, t_k)$ is a collection of k terminal pairs of nodes in G. Flow on an edge $e = (u, v)$ in G can only go from the *tail* u to the *head* v of e. We also assume, to prevent trivial cases, that for each $i = 1, 2, \ldots, k$, there is an s_i-t_i path in G.

- **Attempt #1.** We define $\lambda^*(G, \{(s_i, t_i)\})$ as in the undirected case: it is the supremum of λ such that λ units of flow can be simultaneously sent from each s_i to its corresponding t_i. For a set of nodes $X \subseteq V$, we define $\delta^{out}(X)$ to be the set of all edges $e = (u, v)$ whose tail u is in X and whose head v is in $V - X$. We define $D(X)$ to be the set of all terminal pairs (s_i, t_i) for which $s_i \in X$ and $t_i \notin X$. Finally, we define

$$\eta_1(G, \{(s_i, t_i)\}) = \min_{X \subseteq V} \frac{|\delta^{out}(X)|}{|D(X)|}.$$

- **Attempt #2.** We define $\lambda^*(G, \{(s_i, t_i)\})$ as above. For a set of edges $F \subseteq E$, we define $D'(F)$ to be the set of all terminal pairs (s_i, t_i) for which there is no s_i-t_i path when we delete all the edges in F from G. Finally, we define

$$\eta_2(G, \{(s_i, t_i)\}) = \min_{F \subseteq E} \frac{|F|}{|D'(F)|}.$$

Show that for at least one of these two options ($j = 1$ or $j = 2$), there exist arbitrarily large instances of the directed multicommodity flow problem for which

$$\frac{\eta_j(G, \{(s_i, t_i)\})}{\lambda^*(G, \{(s_i, t_i)\})} = \Omega(k).$$

(2) For a connected graph $G = (V, E)$, let $d(u, v)$, the unweighted distance between nodes $u, v \in V$, be the minimum number of edges in a u-v path. Let $diam(G)$, the diameter of G, be the maximum of $d(u, v)$ over all pairs of nodes u, v.

As in class, let $\lambda_2(G)$ denote the second-smallest eigenvalue of the Laplacian matrix of G. Prove that if G is connected, then

$$diam(G) \geq \frac{1}{n} \cdot \frac{1}{\lambda_2(G)}.$$

(3) Assuming G is a connected graph, how small can $\lambda_2(G)$ be as a function of the number of nodes n?

In particular, let $\overline{\lambda}(n)$ be the minimum value of $\lambda_2(G)$, over all connected n-node graphs G. Find a function $f(\cdot)$ so that for positive constants $c_1 > c_2 > 0$, we have

$$c_1 f(n) \geq \overline{\lambda}(n) \geq c_2 f(n).$$

(4) Suppose we define an analogue of $\lambda_2(G)$ based on absolute values, rather than squaring. For a connected graph G on n nodes, we declare $\mu(G)$ to be the infimum of the ratio

$$\frac{\sum_{(i,j)\in E} |x_i - x_j|}{\sum_{i \in V} |x_i|}$$

over all non-zero vectors $x \in \mathbf{R}^n$ such that $\sum_{i \in V} x_i = 0$.

Give the tightest relationship you can between $\mu(G)$ and the expansion $\alpha(G)$. In particular, find functions $f(\cdot)$ and $g(\cdot)$ that are as "close together" as possible, and show that for all connected graphs G,

$$f(\alpha(G)) \geq \mu(G) \geq g(\alpha(G)).$$

If necessary, your functions f and g can also contain other parameters of the graph G (like its size, or its maximum degree).