CS 683: Advanced Algorithms	**Spectral Partitioning of Planar Graphs**
March 8, 2001	**Scribes: Andrew Hutchings and Mohan Rajagopalan**

1 Bounding the Second Eigenvalue

Last time we showed:

- Every set P of n points in $\mathbb{R}^d$ has a centerpoint x (not neccesarily in P) so that every halfspace through x has $\geq n/(d+1)$ points of P.
- Every planar graph G has a "tangent cap" embedding $C_1, C_2, ..., C_n$ on the sphere so that for some $p_i \in C_i$ the origin is a centerpoint of p_i.

Now we show the following:

Theorem 1.1 *Let G be a planar graph. Then the cut obtained from the eigenvector associated with the 2^{nd} eigenvalue of the Laplacian $L(G)$ has ratio $O(\frac{1}{\sqrt{n}})$.*

Proof. It's enough to show that $\lambda_2 = O(\frac{1}{n})$. To do this, we're going to be using our tangent cap embedding, so we'll need to establish the following:

Lemma 1.2 *For any dimension $d \geq 1$, if $v_1, ..., v_n$ is an L_2 embedding of of G then*

$$\lambda_2 = \min_{\substack{v_1,...,v_n \in \mathbb{R}^d \\ \text{not all 0 s.t} \sum v_i = 0}} \frac{\sum_{ij \in E} \|v_i - v_j\|^2}{\sum_{i \in V} \|v_i\|^2}$$

Proof. $\lambda_2 \geq$ R.H.S, since we could always choose v_i in 1 dimension. So why doesn't λ_2 get smaller? For any embedding with $v_1, ..., v_n$ in $\mathbb{R}^d$, let $v_i = v_{i1}, ..., v_{id}$. Then

$$\frac{\sum_{ij \in E} \|v_i - v_j\|^2}{\sum_{i \in V} \|v_i\|^2} = \frac{\sum_{ij \in E} \sum_{l=1}^{d} (v_{il} - v_{jl})^2}{\sum_{i \in V} \sum_{l=1}^{d} v_{il}^2} = \frac{\sum_{l=1}^{d} \sum_{ij \in E} (v_{il} - v_{jl})^2}{\sum_{l=1}^{d} \sum_{i \in V} v_{il}^2}.$$

But each $\frac{\sum_{ij \in E}(v_{il} - v_{jl})^2}{\sum_{i \in V} v_{il}^2}$ is bounded by λ_2, from what we know about 1 dimensional embeddings. So the whole fraction is bounded by λ_2.

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So we'd like to upper bound $\sum_{ij \in E} \|v_i - v_j\|^2$ and lower bound $\sum_{i \in V} \|v_i\|^2$.

We'll use the "tangent cap" embedding, and take the center of the cap C_i as v_i for node i. Note that the v_i are not the p_i, for which the origin is a centerpoint.

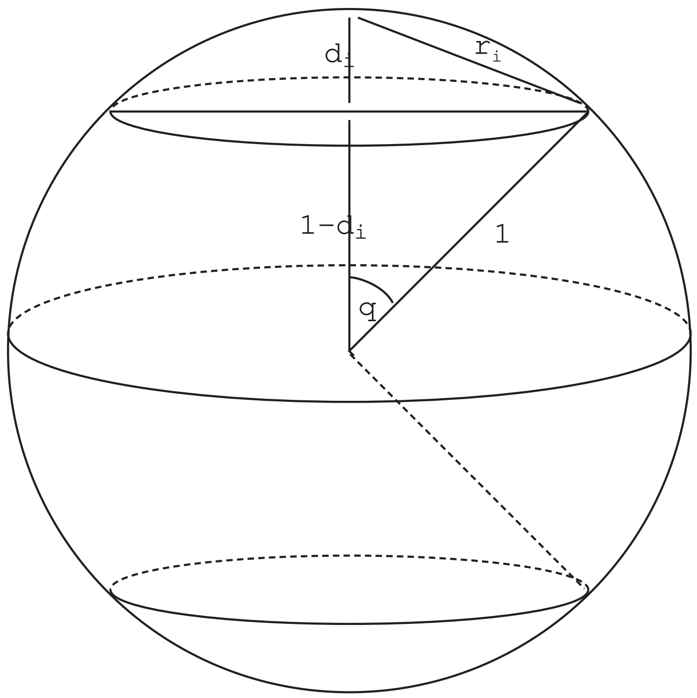

Figure 1: Caps and slices

The upper bound will come essentially from the fact the the sphere has bounded surface area. Let $r_i =$ "diagonal radius" of cap C_i. Then, in Figure 1, $r_i = \sqrt{d_i^2 + 1 - (1-d_i)^2} = \sqrt{2d_i}$. We'll compute the area of C_i using the *Spherical Loaf of Bread Property:* Given a spherical loaf of bread, slices of equal width have equal crust. The area of C_i then must be $\frac{d_i}{2} * 4\pi = 2\pi d_i$. Since $d_i = \frac{1}{2}r_i^2$, the area of C_i is πr^2.

The caps are disjoint, so $\sum_i Area(C_i) = \sum \pi r_i^2 \leq 4\pi$, and so $\sum r_i^2 \leq 4$.

From Figure 2 we can see that $\|v_i - v_j\|^2 \leq (r_i + r_j)^2 = r_i^2 + 2r_ir_j + r_j^2$, which, by the geometric-arithmetic mean inequality, is bounded by $2r_i^2 + 2r_j^2$. Each cap touches $\leq \Delta$ other caps, so

$$\sum_{ij \in E} \|v_i - v_j\|^2 \leq \sum_{ij \in E} 2r_i^2 + 2r_j^2 \leq \sum_i 2\Delta r_i^2 \leq 8\Delta.$$

As for lower bounding $\sum_{i \in V} \|v_i\|^2$, well, notice that this is just n. Thus $\lambda_2 \leq \frac{8\Delta}{n} = O(\frac{1}{n})$

Which is wonderful, except for the fact that $\sum v_i \neq 0$. Hmm... it must be time to use those p_i's we forgot about.

Rotate the sphere so that the centroid m of the v_i's is on the negative z-axis (see Figure 3). Now imagine that each v_i is pinned to m instead of the origin (that is replace each v_i with $v_i - m$). If the origin was a centerpoint of the v_i's, then it must be the case that at least $\frac{n}{4}$ of the v_i's were in the upper hemisphere, so got longer when we pinned them to m. Thus, the denominator $\sum_{i \in V} \|v_i\|^2$ is still $\geq \frac{n}{4}$.

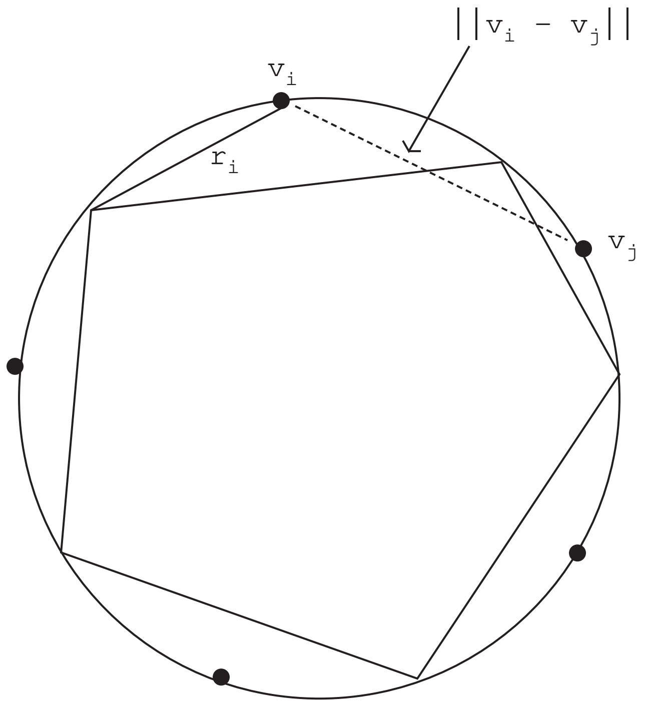

Figure 2: Bounding $\|v_i - v_j\|$

By the above, we know

$$\sum \|p_i - m\|^2 \geq \frac{n}{4}$$

and by the triangle inequality, we have

$$\|v_i - m\| \geq \|p_i - m\| - \|v_i - p_i\|$$

Define

$$J = \{C_i \mid r_i \leq \frac{32}{n}\}$$

$$J' = \{C_i \mid r_i > \frac{32}{n}\}$$

The total area of the caps cannot exceed the area of the sphere, so we have $\sum_{i \in J'} r_i^2 \leq 4$. If $|J'| > \frac{n}{8}$ then we have

$$\sum_{i \in J'} r_i^2 > \frac{32n}{8n} = 4$$

which is a contradiction.

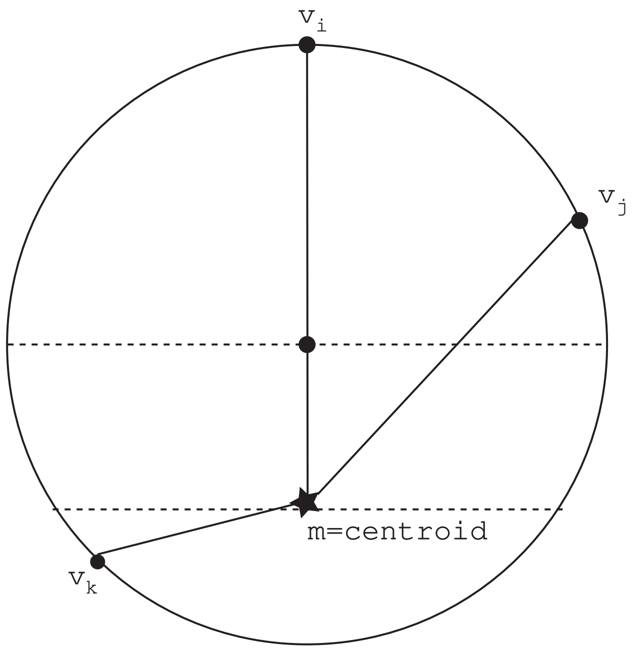

Figure 3: Stretching some v_i's

Therefore assume at least $\frac{n}{8}$ caps in $U \cap J$ and

$$\sum_{i \in V} \|v_i - m\|^2 \geq \sum_{i \in V \cap J} \|v_i - m\|^2 \geq \sum_{i \in V \cap J} \|p_i - m\|^2 - \|v_i - p_i\|^2$$

but $\|p_i - m\| \geq 1$ and $\|v_i - p_i\| \leq \sqrt{\frac{32}{n}} \leq r_i \leq \frac{1}{\sqrt{2}}$, so if we assume $n \geq 64$, since this is all asymptotic in n anyway. Then we have (from above)

$$\sum_{i \in V} \|v_i - m\|^2 \geq \frac{n}{8}(1 - \frac{1}{\sqrt{2}})^2 = \Omega(n).$$

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2 A Variation on the Lipton-Tarjan Theorem

We'll use the above to get an exact bisection of a bounded degree planar graph with only $O(\sqrt{n})$ crossing edges.

The algorithm will maintain "sides" A, $B \in V$ of b_i's initially both empty.

- For $i = 1, 2, \ldots$
- Find S_i, $\bar{S}_i$ of ratio $O(\frac{1}{\sqrt{n_i}})$ where n_i is the current size of graph G.

- Assume $|S_i| \leq |\bar{S}_i|$, and add S_i to the smaller of {A, B }. Delete S_i from G and iterate until G is empty.

Does this process indeed produce a bisection? We note that only the smaller side grows at each step, and at all times:

$$|A| + |B| + |S_i| + |\bar{S}_i| = n$$

Since we are always adding the smaller of A and B to the smaller of S_i and $\bar{S}_i$, the sizes of A and B never overshoot $\frac{n}{2}$, and thus can only yield an exact bisection.

How many crossing edges do we have? At each step we are adding at most $\frac{c}{\sqrt{n_i}}|S_i|$ to the edges crossing the bisection where c is a constant.

In total, this is at most:

$$c \sum_{C_i} \frac{1}{\sqrt{n_i}} |S_i| = c \sum_i \sum_{j \in S_i} \frac{1}{\sqrt{n_i}}$$

Note that we have $|S_i|$ terms for each i in the right-hand sum above, and n terms total. We can clearly upper bound this by the cleaner sum

$$\leq c(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}} + \ldots + \frac{1}{\sqrt{n - |S_1|}} + \frac{1}{\sqrt{n - |S_1| - 1}} + \ldots + 1)$$

Which is simply:

$$\sum_{j=1}^{n} \frac{c}{j^{\frac{1}{2}}} \leq c(1 + \int_1^n \frac{dx}{\sqrt{x}}) = O(\sqrt{n})$$

We can use this sphere embedding to find a separator for planar graphs with arbitrary degrees using $O(\sqrt{n})$ nodes, which is a variation on the following result.

Theorem 2.1 *(Lipton and Tarjan) Any planar graph has a set of nodes X with $|X| \leq 2\sqrt{n}$. Such that the removal of X leaves no component with more than $\frac{3}{4}n$ nodes.*

Proof. Given a sphere embedding of graph G as above, with the origin as the center, choose a hyperplane through the origin by choosing its normal vector uniformly at random. The caps that intersect this hyperplane are a separator since the origin is the center, and we have at most $\frac{3n}{4}$ of the nodes falling on one side of the hyperplane. Now we need only show that number of caps that intersect the hyperplane is small in expectation.

Claim 2.2 *P(Cap C_i hit) $\leq r_i$.*

The set of vectors with normal planes that hit C_i lie in an equatorial band. By the sliced bread theorem, the area of this band is in fact $2 \sin q/2 = \sin q$ (see Figure 1). Calculating $\sin q$ in terms of these radii.

$$\sin q = \sqrt{2d_i - d_i^2} = \sqrt{r_i^2 - \frac{r_i^4}{4}} \leq r_i$$

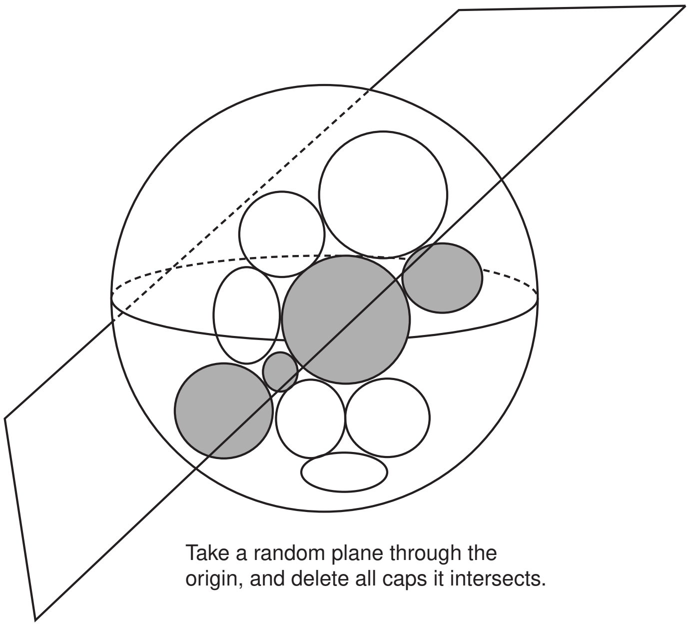

Figure 4: A random hyperplane yields a separator

Claim 2.3 $(\sum_i r_i) \leq 2\sqrt{n}$

We establish this we note

$$(\sum_i r_i)^2 \leq n \sum_i r_i^2 \leq 4n$$

The first inequality is an application of the Cauchy-Schwartz inequality, the second is our previous bound on the cap areas.

If X is the total number of caps hit by the hyperplane, we note that, by linearity of expectation

$$E(X) = \sum_i P(C_i \text{ hit }) \leq \sum_i r_i \leq 2\sqrt{n}$$

Which yields the desired result.

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