CS 683: Advanced Algorithms	Planar Graph partitioning by Eigenvalues
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1 Spectral Partitioning

Last lecture we studied the relation of the second eigenvalue λ_2 to the expansion of the graph. In this lecture we are going to build up the necessary background in order to prove a key theorem in the spectral partitioning of planar graphs.

Spectral partitioning has proved to be an extremely useful heuristic in graph partitioning. It is used in many numerical applications, e.g the physical simulation of surfaces. In simulation, we often need to find to find regions which are relatively independent, so that they can be triangulated and analysed independently. It has been seen that a good partitioning can be obtained on the basis of λ_2, and the availability of fast tools for computing eigenvalues has aided the popularity of this approach. [1] refers to some more of the applications of spectral partitioning.

The theorem that we proved last lecture was

Theorem 1.1 $2\alpha \;\geq\; \lambda_2 \;\geq \frac{\alpha^2}{2\Delta}$

In the context of planar graphs, however, we already have combinatorial guarantees from the following theorem by Lipton and Tarjan (1978).

Theorem 1.2 Planar Seperator Theorem : *Every n-node planar graph has a set of $O(\sqrt{n})$ nodes X so that $G{-}X$ has no component of size greater that $\frac{2n}{3}$. X is called the seperator of the graph.*

In this lecture and the next we shall try to prove this result using eigenvalues. What this means is that we shall try to find whether there is a slice along the λ_2 eigenvector so that we get a seperator of $O(\sqrt{n})$ edges. In doing so, we are slightly distorting the concept of a seperator as stated in the theorem, in that we are actually looking for a edge-seperator rather than a vertex-seperator. As can be seem from the example of a star graph, the Planar Seperator theorem is not true if we naively replace vertex seperators by edge seperators. The problem we encounter is really due to the high degree of a node in the graph. So if we restrict the maximum degree Δ to be $O(1)$, it is possible to get an edge separator of size $O(\Delta\sqrt{n})$, as instead of removing a vertex, we can simply remove all the edges incident to it. We shall in fact establish the stronger theorem.

Theorem 1.3 *Any planar graph G with an even number of nodes and a maximum degree Δ, has a bisection with $O(\sqrt{n}f(\Delta))$ nodes.*

Note that

- Unlike the Leighton-Rao procedure, we are claiming the existence of an exact bisection here with the given size.
- In the proof that we show, the function $f(\Delta)$ will be $O(\Delta^{\frac{3}{2}})$. However the dependence can be made linear by careful analysis.

We begin with a preliminary claim obtained from Theorem 1.2 and Theorem 1.1.

Claim 1.4 *By direct application of Theorem 1.2 and Theorem 1.1, we obtain the ratio of the eigenvector cut to be* $O(\frac{1}{n^{1/4}})$

Proof. From Theorem 1.2, the expansion of a bounded degree connected planar graph $= \alpha = O(\frac{1}{\sqrt{n}})$. And so, $\lambda_2 = O(\frac{1}{\sqrt{n}})$, using Theorem 1.1. So, the eigenvector cut has the ratio $O(\sqrt{2\Delta\lambda_2})$. ∎

Hence, it is obvious that the eigenvector cut does not naively find us something good in this case. To actually show that there is an eigenvector cut of ration $O(\frac{1}{\sqrt{n}})$, we shall have to show that $\lambda_2 = O(\frac{1}{n})$.

Theorem 1.5 Spielman-Tang 1996 *For any bounded degree planar graph,* $\lambda_2 = O(\frac{1}{n})$.

Notes on the theorem

- We shall actually try to find a nice embedding of the graph which wouldgive us a bound on the Raleigh quotient used in defining λ_2.
- The proof of Theorem 1.3 actually follows the bisection procedure (of chiselling away small pieces) used in Leighton-Rao. The interesting point is that here we actually hit the bisection.

With the aim of proving this theorem in view, we digress in order to review some facts from combinatorial geometry.

2 Combinatorial Geometry

Theorem 2.1 (Koebe, Andreev, Thurston). *Let G be a planar graph with vertex set $V = \{1, \ldots, n\}$. Then, there exists a set of disks $\{D_1, \ldots, D_n\}$ in the plane with disjoint interiors such that D_i touches D_j if and only if $(i, j) \in E$.*

This embedding is often referred to as the **tangent-circle** embedding.

Theorem 2.2 *Let $P \subseteq R^d$ be a set of n points. Then there exists a point x in R^d, not necessarily in P, such that, any hyperplane through x divides the point set into two pieces via closed half-spaces, each of size at least $\frac{n}{d+1}$.*

Before proceeding to the proof of this theorem, we shall need to prove three fairly simple and elegant theorems from geometry.

Theorem 2.3 Affine Dependence : *Given nonzero vectors $v_1, v_2, \ldots, v_{d+2} \in R^d$, there exist scalars $\alpha_1, \ldots, \alpha_{d+2}$, not all zero, so that $\Sigma_i \alpha_i v_i = 0$ and $\Sigma_i \alpha_i = 0$.*

Proof. Let $w_1, \ldots w_{d+2}$ be defined as $w_i = \langle v_i, 1\rangle \in R^{d+1}$. So $\{w_i\}$ is linearly dependent. And so there exist a set of scalars $\alpha_1, \ldots, \alpha_{d+2}$, not all zero, so that $\Sigma_i \alpha_i w_i = 0$. These $\{\alpha_i\}$ satisfy the condition in the theorem. ∎

Theorem 2.4 Radon's theorem : *Given $P \subseteq R^d$, $|P| \geq d+2$, there is a partition of P into two sets X and Y such that the convex hulls of X and Y intersect.*

Proof. Let $P = \{p_1, \ldots, p_n\}$ be the set of points where $n \geq d+2$. By the **Affine dependence** theorem, there exist a set of scalars $\{\alpha_i\}$, not all zero, so that $\Sigma_i \alpha_i = 0$ and $\Sigma_i \alpha_i p_i = 0$.

Let $X = \{p_i \mid \alpha_i \geq 0\}$ and $Y = \{p_i \mid \alpha_i < 0\}$. So $\alpha^* = \Sigma_{i \in X} \alpha_i = -\Sigma_{j \in Y} |\alpha_j|$.

Define $p^* = \Sigma_{i \in X} \alpha_i p_i = \Sigma_{j \in Y} |\alpha_j|\ p_j$. Consider the point $\frac{p^*}{\alpha^*} = \Sigma_{i \in X} \frac{\alpha_i}{\alpha^*}\ p_i = \Sigma_{j \in Y} \frac{|\alpha_j|}{\alpha^*}\ p_j$. This point lies in the convex hull of both X and Y by the following fact.

Fact 2.5 *A point q is in the convex hull of $p_1, \ldots, p_n$ iff there exist scalars $\lambda_1, \ldots, \lambda_n$, such that $\forall i\ \lambda_i \geq 0$ and $\Sigma_i \lambda_i = 1$ and $q = \Sigma_i\ \lambda_i p_i$.*

∎

Theorem 2.6 Helly's theorem : *If $C_1, \ldots C_n$ are convex sets in R^d, $n \geq d+1$, and every $d+1$ of these sets have a point in common, then $\bigcap_i C_i \neq \emptyset$.*

Proof. We do the proof by induction. For the case $n = d+1$, the theorem is true.

Let $n \geq d+2$. By induction hypothesis, there exists a point $q_i \in \bigcap_{j \neq i} C_j$ for each $i \in \{1, \ldots, n\}$. Let $Q = \{q_1, \ldots, q_n\}$. Radon's theorem implies a partition of Q into sets X and Y, such that the convex hulls of X and Y intersect and has the point q^*, say. So, for any i, either X or Y is a subset of C_i. Hence, $q^* \in C_i$, since q^* is in convex hulls of both X and Y. So $q^* \in \bigcap_i C_i$. ∎

Finally we prove the Theorem 2.2.

Proof. The idea behind the proof is that we want the centerpoint x to be such that no convex hull of a sufficient fraction ($\geq \frac{d}{d+1}$) of the points misses x. Let $C_1, \ldots, C_N$ be the convex hulls of every set of $\frac{dn}{d+1}$ points. For every $d+1$ of the $C_i's$, fewer than $(d+1)\frac{n}{d+1} = n$ points are left out of their intersection, and so there is least of the $p_i's$ in every set of $d+1$ $C_i's$. Hence, applying Helly's theorem, we get the centerpoint as the intersection of all the $C_i's$. ∎

The following question is open.

Open: There is no known algorithm that is polynomial in n and d for finding a centerpoint.

Later in the course we shall see an algorithm which uses time $O(n + f(d, \epsilon))$ and finds a centerpoint x so that any halfspace through x has at least $(\frac{1}{d+1} - \epsilon)n$ points.

2.1 Miller-Teng-Thurston-Vavasis Embedding

We now consider an embedding of a planar graph on the surface of the unit sphere S^2 in R^3. We define a *cap* on S^2 to be the intersection of S^2 with a plane.

Theorem 2.7 (Miller-Teng-Thurston-Vavasis). *Let G be a planar graph with vertex set $V = \{1, \ldots, n\}$. Then, there exists a set of caps $\{C_1, \ldots, C_n\}$ with disjoint interiors such that C_i touches C_j if and only if $(i, j) \in E$. Moreover, there are points $p_i \in C_i$ such that the origin is a centerpoint of the set $\{p_1, \ldots, p_n\}$/*

This result follows by applying a certain geometric transformation to the tangent-circle embedding of Theorem 2.1.

Definition 2.8 *A circle preserving function $f : R^2 \to S^2$, is one that maps circles in R^2 into the caps on the sphere S^2. The stereographic projection is a circle preserving map in particular.*

Lemma 2.9 *Given $P \subseteq R^2$, there is a circle preserving map of P into the sphere S^2, so that the origin of the sphere is a centerpoint of the image of P.*

To get the Miller embedding, we apply this lemma to the centers of a tangent-circle embedding in R^2. We get the tangent-caps C_i, and the images of the centers are the points $p_i \in C_i$.

References

[1] D. Spielman, S.H. Teng. Spectral Partitioning works: Planar graphs and finite element meshes, *FOCS 1996.*

[2] G.L.Miller, S.H.Teng, W. Thurston, S.Vavasis. Finite element meshes for geometric seperators. *SIAM J. Scientific Computing 1996.*