CS 683: Advanced Algorithms	Graph partitioning: An eigenvalue approach
March 1, 2001	Scribes: Vadim Grinshpun, Mark Sandler

1 Bounds for the eigenvalues of Laplacian matrix

Recall that last time we have introduced the Laplacian matrix L for graph $G = (V, E)$, and started to explore properties of its eigenvalues. We claimed the following inequality:

$$2\alpha \geq \lambda_2 \geq \frac{\alpha^2}{2\Delta},$$

where λ_2 is first non-zero eigenvalue of L. We already proved the upper bound on λ_2, and today we'd like to prove the lower bound. We have also shown that

$$\lambda_2 = \min_x \frac{x^T L x}{x^T x} = \min_x \frac{\sum_E (x_i - x_j)^2}{\sum x_i^2}.$$

where $x \in R^n$, $x \neq 0$, and $\sum x_i = 0$.

Theorem 1.1 *Let x be any vector in R^n s.t. $\sum x_i = 0$. We have*

$$\frac{\sum_E (x_i - x_j)^2}{\sum_i x_i^2} \geq \frac{\alpha^2}{2\Delta}.$$

(As an immediate corollary, we get $\lambda_2 \geq \frac{\alpha^2}{2\Delta}$).

Proof. We will prove an even stronger statement: for any x there exists $s \in R$ such that the cut $(\{x_i | x_i < s\}, \{x_j | x_j \geq S\}$ has ratio $\alpha' \geq \alpha$ s.t. $\frac{\sum_E (x_i - x_j)^2}{\sum_i x_i^2} \geq \frac{\alpha'^2}{2\Delta}$.

Without loss of generality we can assume that $\{x_i\}$ is sorted. Now suppose n is odd, i.e., $n = 2p - 1$. Then x_p is the median coordinate. Translate all the x_i's so that x_p becomes zero, yielding new coordinates $y_i = x_i - x_p$. Clearly $\sum_E (x_i - x_j)^2 = \sum_E (y_i - y_j)^2$. Furthermore, $\sum y_i^2 = \sum (x_i - x_p)^2 = \sum (x_i^2 - 2x_p \sum x_i + n x_p^2)$. But $2x_p \sum x_i = 0$ and $n x_p^2 \geq 0$, so $\sum y_i^2 \geq \sum x_i^2$.

From these, we get the following lower bound:

$$\frac{\sum_E (y_i - y_j)^2}{\sum y_i^2} \leq \frac{\sum_E (x_i - x_j)^2}{\sum x_i^2}.$$

Let E_0 be the set of edges (i, j) that "straddle" p, i.e. where $i < p$ and $j > p$. Replacing edges $(i, j) \in E_0$ with edge pairs (i, p) and (p, j), we get:

$$\sum_E (y_i - y_j)^2 = \sum_{E - E_0} (y_i - y_j)^2 + \sum_{E_0} (y_i - y_j)^2 \leq \sum_{E - E_0} (y_i - y_j)^2 + \sum_{E_0} ((y_i - y_p)^2 + (y_j - y_p)^2)$$

Let $\tilde{E}$ denote $E - E_0$ along with with the subdivided edges in E_0. Then we have

$$\frac{\sum_{E}(y_i - y_j)^2}{\sum y_i^2} \leq \frac{\sum_{\tilde{E}}(y_i - y_j)^2}{\sum y_i^2} = \frac{\sum\limits_{\substack{i,j \leq p \\ (i,j)\in \tilde{E}}} (y_i - y_j)^2 + \sum\limits_{\substack{i,j \geq p \\ (i,j)\in \tilde{E}}} (y_i - y_j)^2}{\sum_{i=1}^{p} y_i^2 + \sum_{j=p}^{y} y_j^2}$$

Here we used the fact that $y_p = 0$ to allow us to insert it into the both sums in the denominator. We can now break our ratio into two parts. [1] Consider the first part (proof for the second part will symmetric). It corresponds to the subgraph induced by nodes 1 through p on graph $(V, \tilde{E})$, which we'll denote as $G_1 = (V_1, E_1)$.
Let d_i^- be the number of edges (j, i) s.t. $i < j$, and d_i^+ the number of edges (i, j) s.t. $i > j$. $\sum_{i=1}^{k} d_i^-$ is a number of edges in the subgraph induced by the first k nodes, and $\sum_{i=1}^{k} d_i^+$ is the number of edges in the subgraph induced by the first k nodes plus the number of edges leaving that subgraph. Therefore, $\sum_{i=1}^{k}(d_i^+ - d_i^-)$ is the number of edges leaving the subgraph induced by first k nodes, or, equivalently, the size of the cut $(\{1 \ldots k\}, \{k+1 \ldots p\})$. This cut naturally corresponds to the cut $(\{1 \ldots k\}, \{k+1 \ldots n\})$ in the original graph G (and has the same number of edges). This means that there are at least $k\alpha$ nodes in it, so $\sum_{i=1}^{k} d_i^+ - d_i^- \geq \alpha k$.
Now we want to estimate our ratio in terms of d_i^- and d_i^+. Note that for every U, V and $\beta > 0$,

$$\beta U^2 + (1/\beta)V^2 \geq 2UV.$$

Adding $U^2 + V^2$ to both sides of inequality yields

$$(1+\beta)U^2 + (1 + 1/\beta^2)V^2 \geq (U+V)^2.$$

Let $U = y_j$ and $V = (y_i - y_j)$ and substitute:

$$y_i^2 \leq (1+\beta)y_j^2 + (1+1/\beta)(y_i - y_j)^2,$$

or

$$(y_j - y_i)^2 \geq \frac{y_i^2 - (1+\beta)y_j^2}{1 + 1/\beta}$$

Summing over all edges in G_1 then we get:

$$\sum_{E_1}(y_i - y_j)^2 \geq \frac{1}{(1+1/\beta)} \sum_{\substack{i<j \\ (i,j)\in E_1}} y_i^2 - (1+\beta)y_j^2 = \frac{1}{(1+1/\beta)} \sum_{i=1}^{p-1} y_i^2 \left(d_i^+ - (1+\beta)d_i^-\right)$$

Thus, our ratio is lower-bounded by

$$\frac{1}{1+1/\beta} \cdot \frac{\sum_{i=1}^{p-1} y_i^2 \left(d_i^+ - (1+\beta)d_i^-\right)}{\sum_{i=1}^{p-1} y_i^2}$$

[1] This is based on the trivial fact that if $\frac{a}{b} > \alpha$ and $\frac{c}{d} > \alpha$ then $\frac{a+c}{b+d} > \alpha$.

Here we sum up to the $p-1$ not to the p, because the last term of the sum is zero anyway, but node p in G_1 may have degree more then Δ.

Before we continue, let us prove a simple lemma about lower-bounding ratios:

Lemma 1.2 *If* $z_1 \geq z_2 \geq \ldots \geq z_m \geq 0$, *and* $\frac{\sum_{i=1}^k \alpha_i}{\sum_{i=1}^k \beta_i} \geq \gamma$ *for* $1 \leq k \leq m$, *then*

$$\frac{\sum_{i=1}^k z_i \alpha_i}{\sum_{i=1}^k z_i \beta_i} \geq \gamma$$

Proof. Notice that

$$\sum_{i=1}^m \alpha_i z_i = \sum_{i=1}^m \alpha_i \sum_{j=i}^m (z_j - z_{j+1}) = \sum_{k=1}^m (z_k - z_{k+1}) \left(\sum_{i=1}^k \alpha_i \right),$$

with $z_{m+1} = 0$, and, analogously

$$\sum_{i=1}^m \beta_i z_i = \sum_{k=1}^m (z_k - z_{k+1}) \left(\sum_{i=1}^k \beta_i \right),$$

but

$$\frac{(z_k - z_{k+1}) \left(\sum_{i=1}^k \alpha_i \right)}{(z_k - z_{k+1}) \left(\sum_{i=1}^k \beta_i \right)} = \frac{\sum_{i=1}^k \alpha_i}{\sum_{i=1}^k \beta_i} \geq \gamma,$$

and since $a_i/b_i \geq \gamma$ implies $\sum a_i / \sum b_i \geq \gamma$, it immediately follows that

$$\frac{\sum_{i=1}^k z_i \alpha_i}{\sum_{i=1}^k z_i \beta_i} \geq \gamma$$

■

Now, take $z_i = y_i^2$, $\alpha_i = d_i^+ - (1+\beta) d_i^-$ and $\beta_i = 1$. Then we would have:

$$\sum_{i=1}^k \alpha_i / \beta_i = \frac{\sum_{i=`}^k d_i^+ - (1+\beta) d_i^-}{k} = \frac{\sum_{i=1}^k (d_i^+ - d_i^-) - \beta \sum_{i=1}^k (d_i^-)}{k} \geq \frac{\alpha k - \beta \frac{\Delta k - \alpha k}{2}}{k} = \alpha - \beta \frac{\Delta - \alpha}{2}$$

Here we have used the fact that in a subgraph of size k there are at most $\frac{\Delta k - \alpha k}{2}$ edges, where Δ is maximal degree of node. Now choose $\beta = \frac{\alpha}{\Delta - \alpha}$ then we have

$$\sum_{i=1}^k \alpha_i / \beta_i \geq \alpha - \beta \frac{\Delta - \alpha}{2} = \alpha k - \frac{\alpha}{2} = \frac{\alpha}{2}$$

Therefore by applying the lemma above we acquire the following result

$$\frac{\sum_{i=1}^{p-1} y_i^2 (d_i^+ - (1+\beta) d_i^-}{\sum_{i=1}^{p-1} y_i^2} \geq \frac{\alpha}{2}$$

and therefore

$$\frac{1}{1+1/\beta} \cdot \frac{\sum_{i=1}^{p-1} y_i^2 (d_i^+ - (1+\beta) d_i^-)}{\sum_{i=1}^{p-1} y_i^2} \geq \frac{\alpha}{\Delta} \cdot \frac{\alpha}{2}.$$

Recalling that first expression in this sequence was a lower bound on our ratio, we get that

$$\frac{\sum_{E_1} (y_i - y_j)^2}{\sum y_i^2} \geq \frac{\alpha^2}{\Delta}$$

Analogously for the second part we have

$$\frac{\sum_{E_2} (y_i - y_j)^2}{\sum y_i^2} \geq \frac{\alpha^2}{\Delta},$$

and this completes the proof. ■