CS 683: Advanced Algorithms	Lecture 9: Bourgain's Theorem
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1 Introduction

In this lecture we address the question: By how much can the narrowest cut value $\eta(G, \{s, t\})$ exceed the maximum concurrent flow value $\lambda^*(G, \{s, t\})$ for a general graph G?

We saw from expander graphs with all-pairs as terminals that η/λ^* can be $\Omega(\log n)$, and from the Leighton-Rao Theorem we know that η/λ^* is $O(\log n)$ for general graphs with all-pairs as terminals. Also from last time we know that for planar graphs with all terminals on a single face $\eta/\lambda^* = 1$. What we will do today is to generalize the Leighton-Rao Theorem by showing that $\eta/\lambda^* \leq O(\log n)$ for general graphs with arbitrary placement of terminals. This result was shown independently by Linial-London-Rabinovich (1995) and by Aumann-Rabanni (1998). In a sense, this is the theorem that we really want, with the Leighton-Rao Theorem being a good stepping stone to it. This result is based on the theory of metric embeddings and we shall prove it by first relating the ratio η/λ^* to metric embeddings and then proving Bourgain's Theorem (1985) on metric embeddings.

2 Metric Embeddings

2.1 First Lemma

Definition 2.1 *A <u>(finite) metric space</u> is a set V of n points, and a distance function d, defined as $d : V \times V \to \mathbf{R}$, and satisfying the metric properties:*

- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) \geq 0$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

In the proof here we will be working with pseudo-metrics where the first property is weakened to $x = y \Rightarrow d(x, y) = 0$. An important example of a finite metric space is the shortest-path metric where for a graph with non-negative lengths on its edges we define $d(u, v)$ to be the length of the shortest $u - v$ path.

Lemma 2.2

$$\lambda^*(G, \{s, t\}) = \min_{\substack{metrics\ d \\ (where\ denom.>0)}} \frac{\sum_{(u,v)\in E} d(u, v)}{\sum_{i=1}^{k} d(s_i, t_i)}$$

Proof.

($\geq$) The optimal dual solution $\{y_e\}$ defines a shortest-path metric and

$$\min_{\substack{metrics\ d \\ (where\ denom.>0)}} \frac{\sum_{(u,v)\in E} d(u,v)}{\sum_{i=1}^{k} d(s_i,t_i)} \leq \frac{\sum_{e\in E} y_e}{1} = \lambda^*$$

($\leq$) We start with a metric d. This defines edge lengths where for $e = (u,v)$, $y_e = d(u,v)$. By the triangle inequality $d(s_i,t_i) \leq$ shortest-path length from s_i to t_i with edge lengths y_e. We then scale edge lengths by

$$\frac{1}{\sum_{i=1}^{k} \text{shortest-path-dist}(s_i,t_i)}.$$

and it is easy to verify that the resulting edge lengths satisfy the requirements of the dual. Now from LP duality,

$$\lambda^* \leq \frac{\sum_{(u,v)\in E} d(u,v)}{\sum_{i=1}^{k} \text{shortest-path-dist}(s_i,t_i)} \leq \frac{\sum_{(u,v)\in E} d(u,v)}{\sum_{i=1}^{k} d(s_i,t_i)}$$

■

2.2 Second Lemma

Definition 2.3 *A partition (cut) $S, V\setminus S$ defines a metric*

$$\Delta_S(x,y) = \begin{cases} 1 & if\ |S \cap \{x,y\}| = 1 \\ 0 & otherwise \end{cases}.$$

A cut metric is a metric arising in this way.

Given metrics d_1, d_2, scalars $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 d_1 + \alpha_2 d_2$ is a also a metric. We can now ask the question: What metrics arise as non-negative combinations of cut metrics?

Definition 2.4 *The L_1 metric is defined as*

$$L_1(<x_1, x_2, \cdots, x_l>, <y_1, y_2, \cdots, y_l>) = \sum_{i=1}^{l} |x_i - y_i|.$$

Claim 2.5 *Every combination of cut metrics can be realized by points in* $\mathbf{R}^l$ *with the* L_1 *metric, for some* l.

For $\alpha_1\Delta_{S_1} + \alpha_2\Delta_{S_2}$, map $x \in V$ to $<x_1, x_2>$, where

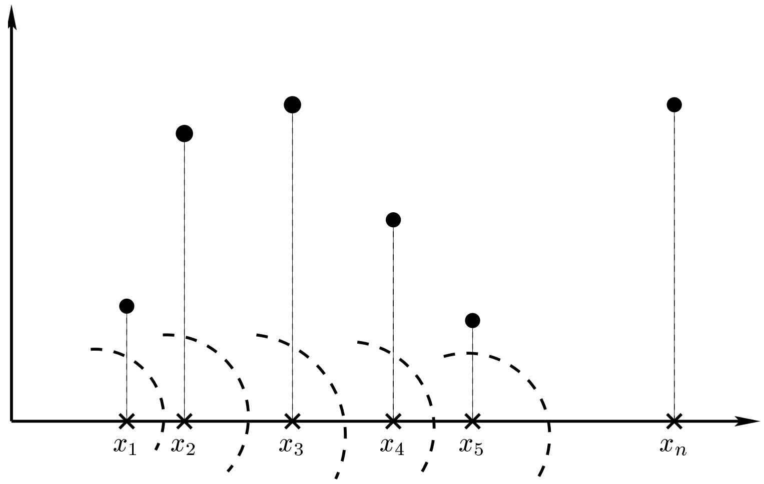
$$x_1 = \begin{cases} 0 & if \;\; x \in S_1 \\ \alpha_1 & if \;\; x \notin S_1 \end{cases}, \qquad \text{and } x_2 = \begin{cases} 0 & if \;\; x \in S_2 \\ \alpha_2 & if \;\; x \notin S_2 \end{cases}.$$

Definition 2.6 *A metric space* (V, d) *is* <u>L_1*-representable*</u> *if* $V \subseteq \mathbf{R}^l$ *and* d *is the* L_1 *metric on them.*

Conversely, we have:

Claim 2.7 *If* $V \subseteq \mathbf{R}^l$ *is a set of* n *points, and* d *is the* L_1 *metric on them, then the metric space* (V, d) *is actually a non-negative combination of cut metrics.*

Proof.



We proceed by first projecting the points into each dimension separately, generating appropriate cut metrics for each dimension and then at the end putting all the cuts representing each dimension together. Say, in a single dimension, the points have coordinates $x_1 \leq x_2 \leq \cdots \leq x_n$ (assume this indexing). Define $S_i = \{1, 2, \cdots, i\}$ for $i = 1, 2, \cdots, n-1$. Define $\alpha_i = x_{i+1} - x_i \geq 0$. It is easy to confirm then that the distance in this dimension is $\sum_{i=1}^{n-1} \alpha_i \Delta_{S_i}$. ■

Lemma 2.8

$$\eta(G, s, t) = \min_{\substack{L_1-representable \\ metrics\ d}} \frac{\sum_{(u,v)\in E} d(u, v)}{\sum_{i=1}^{k} d(s_i, t_i)}$$

Proof.

($\geq$) Consider the narrowest cut S. We have:

$$\eta = \frac{|\delta(S)|}{|D(S)|}, \quad |\delta(S)| = \sum_{uv \in E} \Delta_S(u,v), \quad |D(S)| = \sum_{i=1}^{k} \Delta_S(s_i, t_i),$$

and Δ_S is L_1-representable.

($\leq$) Let d be any L_1-representable metric $d = \sum_j \alpha_j \Delta_{S_j}$. We have:

$$\frac{\sum_{(u,v)\in E} d(u,v)}{\sum_{i=1}^{k} d(s_i,t_i)} = \frac{\sum_{(u,v)\in E} \sum_j \alpha_j \Delta_{S_j}(u,v)}{\sum_{i=1}^{k} \sum_j \alpha_j \Delta_{S_j}(s_i,t_i)} = \frac{\sum_j \alpha_j \sum_{(u,v)\in E} \Delta_{S_j}(u,v)}{\sum_j \alpha_j \sum_{i=1}^{k} \Delta_{S_j}(s_i,t_i)} =$$

$$\frac{\sum_j \alpha_j\, |\delta(S_j)|}{\sum_j \alpha_j\, |D(S_j)|} \geq \min_j \frac{\cancel{\alpha_j}\, |\delta(S_j)|}{\cancel{\alpha_j}\, |D(S_j)|} \geq \eta.$$

On the second-last step we used the fact that for reals $a_1, a_2, \cdots, a_n \geq 0, b_1, b_2, \cdots, b_n > 0$, we have $\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \geq \min_{i=1,2,\cdots,n} \frac{a_i}{b_i}$.

■

3 Bourgain's Theorem on Metric Embeddings

Definition 3.1 *An embedding of finite metric space (V, d) into (not necessarily finite) metric space (V', d') is a function $f : V \to V'$.*

Definition 3.2 *A metric space (V, d) is embeddable in (V', d') with distortion $\gamma \geq 1$ if $\forall x, y \in V : d(x, y) \leq d'(f(x), f(y)) \leq \gamma\, d(x, y)$.*

An alternative definition would be: (V, d) is embeddable in (V', d') with distortion $\gamma \geq 1$ if $\exists \beta > 0$ so that $\forall x, y \in V : \beta d(x, y) \leq d'(f(x), f(y)) \leq \gamma\beta d(x, y)$. As a special case, we have:

Definition 3.3 *(V, d) is isometrically embeddable in (V', d') if it is embeddable with distortion 1.*

Suppose that every n-point metric space (V, d) were embeddable in $(\mathbf{R}^l, L_1)$, (l could be up to 2^n) with distortion $\leq \gamma_n$ (for some constant γ_n depending on n), then we have the following:

Claim 3.4 *$\eta/\lambda^* \le \gamma_n$, for all instances.*

Proof. We take the optimal dual solution $\{y_e\}$, construct the shortest-path metric d from it and embed (V, d) in $(\mathbf{R}^l, L_1)$ with distortion $\le \gamma_n$ (using the first definition.) So

$$\lambda^* = \frac{\sum\limits_{(u,v)\in E} d(u,v)}{\sum\limits_i d(s_i,t_i)} \ge^1 \frac{\frac{1}{\gamma_n}\sum\limits_{(u,v)\in E} L_1(f(u),f(v))}{\sum\limits_i d(s_i,t_i)} \ge^2 \frac{\frac{1}{\gamma_n}\sum\limits_{(u,v)\in E} L_1(f(u),f(v))}{1\times\sum\limits_{i=1}^{k} L_1(f(s_i),f(t_i))} \ge \frac{1}{\gamma_n}\eta.$$

■

Theorem 3.5 *(Bourgain's Theorem): Every n-point metric space is embeddable in $(\mathbf{R}^l, L_1)$ (for some l) with distortion $O(\log n)$.*

Proof. Next time ■

This will also give us an approximation algorithm for η:

- Take the optimal dual solution and construct d from it.
- Apply a constructive version of Bourgain's Theorem to embed d as d'.
- Represent d' as a combination of cuts.
- Finally, take the best of these cuts.

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[1]this uses the $d'(f(x), f(y)) \le \gamma\, d(x, y)$ part of definition 3.2
[2]this uses the $d(x, y) \le d'(f(x), f(y))$ part of definition 3.2