CS 683: Advanced Algorithms	**Okamura–Seymour Theorem (1981)**
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1 Okamura–Seymour Theorem

In this lecture, we will prove the following theorem about a special class of multicommodity-flow-problem instances. The proof yields an algorithm for such instances. Recall that, for a graph $G = (V, E)$, terminal node pairs $\{(s_i, t_i)\}_{i=1}^k$, and $X \subseteq V$, $\delta(X)$ is the set of edges with one end in X and the other in $V \setminus X$; and $D(X)$ is the set of terminal pairs with one end in X, the other in $V \setminus X$.

Theorem 1.1 (Okamura–Seymour (1981)) *Let $(G, \{(s_i, t_i)\}_{i=1}^k)$ be an instance of multicommodity flow problem such that G is planar with all terminals on a single face (w.l.o.g., outer face). Suppose the cut condition and the evenness condition hold:*

$$\forall X \subseteq V, \ \ |\delta(X)| - |D(X)| \geq 0 \text{ and even.}$$

Then there exist edge-disjoint paths $P_1, \ldots, P_k$ such that P_i links s_i to t_i.

Definition 1.2 *Suppose the cut condition holds. We define a set X to be* tight *if*

$$|\delta(X)| = |D(X)|.$$

Recall the fact from the last lecture that if both X and Y are tight, and there is no demand with one end in $X \setminus Y$, the other in $Y \setminus X$ then $X \cap Y$ and $X \cup Y$ are both tight and $e(X \setminus Y, Y \setminus X) = \emptyset$.

Proof. The proof of the theorem will proceed by induction on the size of the edge set, E.

For the rest of this discussion, we assume that the outer face of G is a simple cycle, which we call C. If we define node v to be a *cut-node* when its deletion disconnects the graph, our assumption is equivalent to G having no cut-nodes.

The reason we can make this assumption without loss of generality is depicted in Figure 1. Suppose v is a cut-node in graph G, deletion of which disconnects the graph into two connected components, G_1 and G_2. In this case, we can generate two separate problem instances such that the solutions to these "subproblems" could be combined to obtain a solution to the original problem. These subproblems are defined on subgraphs G_1 and G_2 (with inclusion of v in both of them), respectively. Any terminal pair with both ends in the same component is included only in the corresponding subproblem. If a terminal pair (s_l, t_l) is split between G_1 and G_2 then in each subproblem, node v plays the role of the missing component of the terminal pair. It is easy to see that the solutions to the subproblems can be easily combined to obtain a solution to the original problem, because these "split" terminal

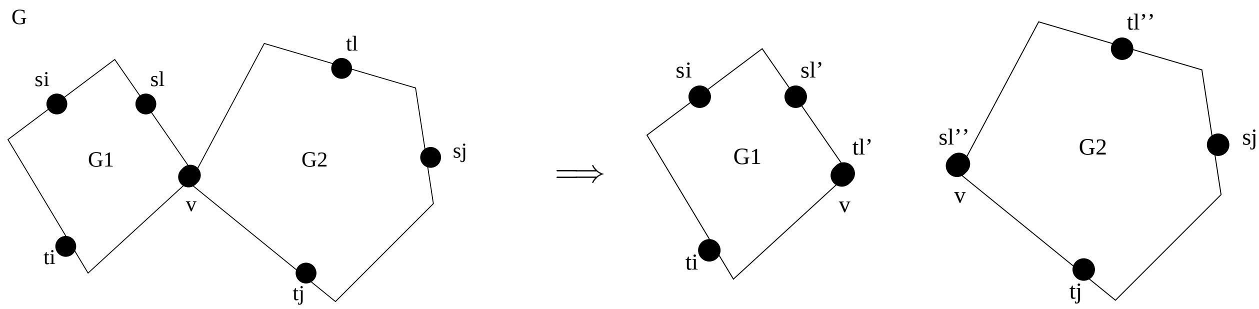

Figure 1: Cut-node elimination

pairs are required to be connected via the cut-node v in the original problem. It could be shown that induction hypothesis applies to these subproblems.

Our induction strategy is as follows: We carefully choose an edge $e = (a, b)$ on the cycle C and a terminal pair (s_j, t_j). After removing e from the graph, we replace the terminal pair (s_j, t_j) with two terminal pairs (s_j, a) and (b, t_j). At this point, our choice of e and (s_j, t_j) makes the induction hypothesis applicable. Algorithmically, our strategy could be viewed as committing edge e to the link between (s_j, t_j) and linking s_j to a and b to t_j to use $e = (a, b)$ later to complete this link. Now we need to show that we can choose such an edge and terminal pair.

Our worry is that when remove the edge e and modify terminal pairs, we may violate the cut condition or the evenness condition. Note that any set X such that $\delta(X) \cap C = \emptyset$ will maintain both of these conditions. So, we characterize the sets we need to focus on as follows.

Definition 1.3 *Define set X to be* critical *if it is tight and $\delta(X)$ contains exactly two edges of the cycle C.*

Once we choose our edge e and the terminal pair (s_j, t_j), three "types" of critical sets, which are depicted in Figure 2, can cause problems, i.e., could violate either the cut condition or the evenness condition. If we call the value $|\delta(X)| - |D(X)|$ as the surplus of cut X, then, by inspection, we can see that our modification reduces the surplus of a cut X by an even number that is at most 3. So, surplus drops by 2 if it drops at all.

The following lemma shows that we can choose the edge and the terminal pair to keep the surplus of any such critical set from dropping. This concludes the proof of the theorem.

Lemma 1.4 *There exist an edge $e = (a, b)$ and a terminal pair (s_j, t_j) so that there is no critical set X with $X \cap \{s_j, t_j\} = \emptyset$ but $X \cap \{a, b\} \neq \emptyset$.*

If this were true we can choose this e and (s_i, t_i) use induction strategy, and no critical set can drop to surplus -2. So, induction hypothesis is applicable once we make the modification to the graph mentioned above.

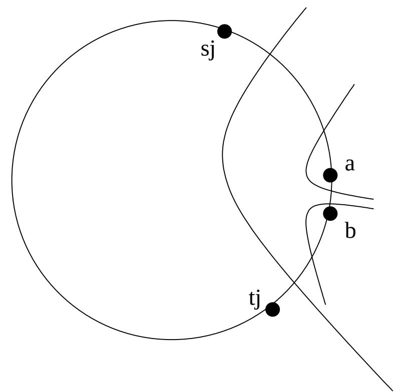

Figure 2: Potentially problematic critical sets

Proof. Assume there are some critical sets (if not, we are done). Choose the critical set X so that $|X \cap C|$ is as small as possible. Define edge $e = (a, b)$ to be one of the two edges of C in $\delta(X)$.

Of all terminal pairs in $D(X)$, choose the one with a terminal clockwise as close to b as possible, say j. By symmetry, we choose $s_j \in X, t_j \notin X$.

Suppose, by way of contradiction, there exists critical set X' such that $X' \cap \{s_j, t_j\} = \emptyset$ but $X' \cap \{a, b\} \neq \emptyset$. Then we claim the following.

Claim 1.5 *No terminal pair has one end in $X \setminus X'$ and the other in $X' \setminus X$.*

Proof. Such a pair have one end in X, the other end in X' closer to b than t_j is, contradicting our choice of (s_j, t_j). ■

Since both X and X' are tight and $D(X \setminus X', X' \setminus X) = \emptyset$, we know that $X \cap X'$ is tight and $e(X \setminus X', X' \setminus X) = \emptyset$ by the fact proven in the last lecture. In particular $e = (a, b) \notin e(X \setminus X', X' \setminus X)$, so $(a, b) \notin \delta(X)$. But at least one of a or b is in X'. So both are. Therefore, $a \in X \cap X'$.

Since $X \cap X'$ is nonempty, critical and $s_j \notin X \cap X'$ (since we assume that $s_j \notin X'$), $|(X \cap X') \cap C| < |X \cap C|$. This contradicts our initial assumption about the minimality of $|X \cap C|$ and concludes the proof. ■

Theorem ■