CS 683: Advanced Algorithms	The Okamura-Seymour Theorem I
February 13, 2001	Scribe: Tim Roughgarden

1 Introduction

Thus far we have investigated the relationship between (fractional) multicommodity flows and sparse (or "narrow") cuts only when each pair of nodes represents a commodity. Today we continue to investigate this relationship, in the more general setting where we are given an undirected graph $G = (V, E)$ and k commodities corresponding to source-sink pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$.[1]

As usual, we denote by $\lambda^* = \lambda^*(G, \{(s_i, t_i)\})$ the largest value λ of flow that be shipped simultaneously between each s_i-t_i pair (respecting edge capacities), and by $\eta = \eta(G, \{s_i, t_i\})$ the value of the sparsest cut in G:

$$\min_{U \subseteq V} \frac{|e(U, V \setminus U)|}{d(U, V \setminus U)}.$$

(Here $e(U, V\setminus U)$ is the set of edges with precisely one endpoint in U and $d(U, V\setminus U)$ is the number of commodities with precisely one terminal in U.) Recall that for any instance we have the inequality $\lambda^*(G, \{(s_i, t_i)\}) \leq \eta(G, \{(s_i, t_i)\})$ and that Leighton and Rao proved a relaxation of the reverse inequality, namely $\eta(G, \{(s_i, t_i)\}) \leq O(\log n) \cdot \lambda^*(G, \{s_i, t_i\})$ (where $n = |V|$), in the special case where each pair of vertices is an s_i-t_i pair. A natural extension of the Leighton-Rao theorem to hope for is an inequality of the type

$$\eta(G, \{(s_i, t_i)\}) \leq f(n, k) \cdot \lambda^*(G, \{(s_i, t_i)\})$$

for *any* multicommodity flow instance and some "reasonable" function $f(n, k)$ (ideally, $O(\log n)$ or $O(\log k)$). We will return to this question in future lectures, but today we will explore a different (though related) direction. Namely, what kinds of hypotheses on G and the location of the terminal pairs ensure that

$$\eta(G, \{(s_i, t_i)\}) = \lambda^*(G, \{(s_i, t_i)\});$$

equivalently, can we find a reasonably rich collection of instances for which the cut condition[2] is both necessary and sufficient for the existence of a concurrent flow of value 1?

2 The Okamura-Seymour Theorems

First, note that there a few special cases for which it's easy to prove that $\lambda^* = \eta$:

[1] We allow distinct commodities to share sources, sinks, or both.

[2] Recall the *cut condition* holds when for all $U \subseteq V$, $d(U, V\setminus U) \leq |e(U, V\setminus U)|$.

(1) when $k = 1$ – i.e., when there's only one commodity (Proof: equivalent to the max-flow/min-cut theorem.)

(2) when all sources (equivalently, all sinks) are identical (Proof: add a supersink t, attach all sinks t_i to t by an arc with infinite capacity, and apply the max-flow/min-cut theorem.)

(3) when G is a tree (Proof: route $\eta(G, \{(s_i, t_i)\})$ units of flow from each s_i to each t_i on the unique tree path between them, yielding a concurrent flow f. To finish, we need only note that f is feasible: an edge e induces a cut $(U, V\backslash U)$ for which $e(U, V\backslash U) = \{e\}$, and thus e will carry $d(U, V\backslash U) \cdot \eta(G, \{(s_i, t_i)\}) \leq 1$ units of flow in f.)

Here are two more cases where $\lambda^* = \eta$, though the proofs are harder:

(4) when G is a cycle (this will follow from the Okamura-Seymour Theorem, below)

(5) when $k = 2$ (this is proved using somewhat different techniques)

Okamura and Seymour proved that $\lambda^* = \eta$ for a fairly wide class of graphs, as follows.

Theorem 2.1 (Okamura-Seymour '81) *Suppose $G = (V, E)$ is a planar graph with k terminal pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$. Suppose further that all of the terminals lie on the same face in some planar drawing of G. Associate a capacity $c_e \geq 0$ with each edge e and a demand $d_i \geq 0$ with each source-sink pair (s_i, t_i). Then*

$$\max\{\lambda : \exists \textit{ feasible flow of value } \lambda\} = \min_{U \subseteq V} \frac{c(U, V\backslash U)}{d(U, V\backslash U)}.$$

We have introduced some new notation for instances with non-unit demands and capacities. First, a feasible flow of value λ is one that routes λd_i units of flow from s_i to t_i concurrently for all i. Also, $c(U, V\backslash U)$ is the sum of capacities of edges in $e(U, V\backslash U)$ while $d(U, V\backslash U)$ is the sum of demands of commodities with precisely one terminal in U (note this agrees with our old definition in the case of unit demands).

Before we plunge into the proof, we mention two applications of the Okamura-Seymour theorem (as the hypotheses may seem strange and/or overly restrictive). First, the theorem includes other natural special cases, including the cases where G is a tree or a cycle (these graphs are clearly planar, and every vertex is on every face in any planar drawing). Second, in practice these types of routing problems arise in the context of *VLSI design*, where the hypotheses are often satisfied. For example, if one wants edge-disjoint connections from a bunch of pins on one side of a chip to a bunch of pins on another side of a chip, we have exactly the sort of problem covered by the Okamura-Seymour Theorem.[3]

In the next lecture, we will prove the Okamura-Seymour Theorem in the following slightly stronger form.

[3] Of course, in this context we would like to have all commodities routed *unsplittably* rather than fractionally; more on this later.

Theorem 2.2 (Okamura-Seymour '81) *Suppose $G = (V, E)$ is a planar graph with k terminal pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$. Suppose further that all of the terminals lie on the same face in some planar drawing of G, and that the* evenness condition *holds: $|e(U, V \backslash U)| + d(U, V \backslash U)$ is even for all $U \subseteq V$. If the cut condition holds, then there are edge-disjoint paths $P_1, \ldots, P_k$ such that P_i is an s_i-t_i path for each i.*

It is not obvious that this theorem implies Theorem 2.1; this reduction is our next order of business. Indeed, the two theorems should feel somewhat incomparable (the first allows for arbitrary demands and capacities, while the second instead insists that flow is routed unsplittably).

The reduction of Theorem 2.1 to Theorem 2.2 proceeds in 6 steps:

(1) Theorem 2.1 is true if and only if it is true for instances with rational demands and capacities (if there is a counterexample with irrational data, a "nearby" instance with rational data also provides a counterexample).

(2) Theorem 2.1 is true if and only if it is true for instances with integral demands and capacities (an instance with arbitrary rational data is equivalent to the one obtained by clearing all denominators).

(3) Theorem 2.1 is true if and only if it is true for instances with unit demands and capacities (integral capacities may be replaced by parallel edges and commodities with integral demands may by split into many commodities with unit demand sharing the same source and sink). If you prefer simple graphs, subdivide each copy of a parallel edge.[4]

(4) We may strengthen the theorem by asserting (for instances with unit demands and capacities) the existence of a *half-integral* flow (i.e., a flow in which each flow path carries either $\frac{1}{2}$ or 1 unit of flow), rather than merely an arbitrary fractional flow.

(5) The statement of step (4) is equivalent to the existence of edge-disjoint paths $P_1, \ldots, P_k$ (P_i an s_i-t_i path) under the additional hypotheses that all edges and demands come in identical pairs (by replicating edges and commodities as in step (3)).

(6) Theorem 2.2 is a generalization of the statement of step (5) (if all edges and demands come in pairs, then $|e(U, V \backslash U)|$ and $d(U, V \backslash U)$ are always even and hence so is their sum).

Notice that at no point in the reduction do we destroy either planarity or the property that G can be drawn with all terminals on a common face.

[4]If you're thinking algorithmically, this step should bother you as we may have blown up the size of the problem tremendously. There are two pieces of good news: first, this reduction can be done more efficiently and second, the VLSI problem mentioned earlier typically has unit demands and capacities, anyway.

3 Four Lemmas about the Cut and Demand Functions

In preparation for the proof of the Okamura-Seymour Theorem, we need to prove that the functions $|e(\cdot,\cdot)|$ and $d(\cdot,\cdot)$ have several nice properties. For brevity, we write $\delta(X)$ for $e(X, V\backslash X)$ and $D(X)$ for the set of commodities with precisely one terminal in X (and we write $\delta(x), D(x)$ instead of $\delta(\{x\}), D(\{x\})$). The first lemma says that checking the evenness condition (i.e., that $|\delta(X)|$ and $|D(X)|$ have the same parity for all $X \subseteq V$) reduces to checking it only for singleton sets.

Lemma 3.1 *$|\delta(X)| - |D(X)|$ is even for all $X \subseteq V$ if and only of $|\delta(x)| - |D(x)|$ is even for all $x \in V$.*

Proof. One direction is trivial; for the other, suppose that $|\delta(x)| - |D(x)|$ is even for all $x \in V$ and fix any $X \subseteq V$. Consider the sums $\sum_{x\in X} |\delta(x)|$ and $\sum_{x\in X} |D(x)|$. By assumption, $|\delta(x)|$ and $|D(x)|$ have the same parity for each $x \in X$, so the two sums have the same parity. On the other hand, the first sum overestimates $\delta(X)$ by precisely twice the number of edges with both endpoints in X (an even number) while the second sum overestimates $D(X)$ by twice the number of commodities with both terminals in X (again, an even number). Thus, $|\delta(X)|$ and $|D(X)|$ have the same parity. ∎

The second lemma states that to check the cut condition for a graph, it suffices to check only vertex sets for which the set and its complement induce connected graphs.

Lemma 3.2 *Let $G = (V, E)$ be a connected graph. If for some $X \subseteq V$ we have $|\delta(X)| < |D(X)|$, there exists an X' satisfying $|\delta(X')| < |D(X')|$ such that $G[X']$ and $G[V\backslash X']$ are connected graphs.*

Proof. Suppose G is connected and the cut condition fails. Choose X with $|\delta(X)| < |D(X)|$ minimizing $|\delta(X)|$; we will show that both $G[X]$ and $G[V\backslash X]$ are connected. Suppose otherwise, and assume without loss of generality that $G[X]$ is disconnected.[5] Since $G[X]$ is disconnected, we may write X as the disjoint union of non-empty X_1, X_2 with no edges having one endpoint in each of X_1, X_2. Since G is connected and $\delta(X_1) \cap \delta(X_2) = \emptyset$, $\delta(X_1), \delta(X_2)$ is a non-trivial partition of $\delta(X)$ (see Figure 1). Since $|\delta(X_i)| < |\delta(X)|$ for $i = 1, 2$, our choice of X ensures that the cut condition holds for X_1, X_2. Putting it all together, we have

$$|D(X_1)| + |D(X_2)| \leq |\delta(X_1)| + |\delta(X_2)| = |\delta(X)| < |D(X)| \leq |D(X_1)| + |D(X_2)|,$$

providing the desired contradiction. ∎

The third lemma investigates the relationship between two cuts, their union, and their intersection.

[5]Since $\delta(X) = \delta(V\backslash X)$, we may exchange the roles of X and $V\backslash X$, if necessary.

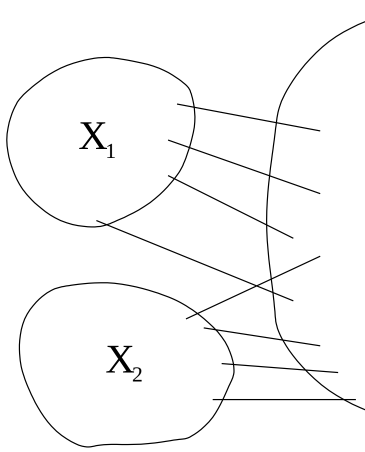

Figure 1: Proof of Lemma 3.2

Lemma 3.3 *Let $G = (V, E)$ be a graph and $X, Y \subseteq V$. If there are p edges with one endpoint in each of $X \backslash Y, Y \backslash X$, then*

$$|\delta(X)| + |\delta(Y)| = |\delta(X \cup Y)| + |\delta(X \cap Y)| + 2p.$$

Proof. This is a simple (if somewhat tedious) counting argument (see Figure 2):

(1) edges with one endpoint in each of $X \cap Y$, $V \backslash (X \cup Y)$ contribute once to each $\delta(X)$, $\delta(Y)$, $\delta(X \cup Y)$, $\delta(X \cap Y)$

(2) edges with one endpoint in each of $X \backslash Y$, $V \backslash (X \cup Y)$ contribute once to each of $\delta(X)$, $\delta(X \cup Y)$

(3) edges with one endpoint in each of $Y \backslash X$, $V \backslash (X \cup Y)$ contribute once to each of $\delta(Y)$, $\delta(X \cup Y)$

(4) edges with one endpoint in each of $X \backslash Y$, $X \cap Y$ contribute once to each of $\delta(Y)$, $\delta(X \cap Y)$

(5) edges with one endpoint in each of $Y \backslash X$, $X \cap Y$ contribute once to each of $\delta(X)$, $\delta(X \cap Y)$

(6) edges with one endpoint in each of $Y \backslash X$, $X \backslash Y$ contribute once to each of $\delta(X)$, $\delta(Y)$.

■

In particular, $|\delta(\cdot)|$ is a *submodular* function on V, which is to say that $|\delta(X)| + |\delta(Y)| \geq |\delta(X \cup Y)| + |\delta(X \cap Y)|$ for all $X, Y \subseteq V$.

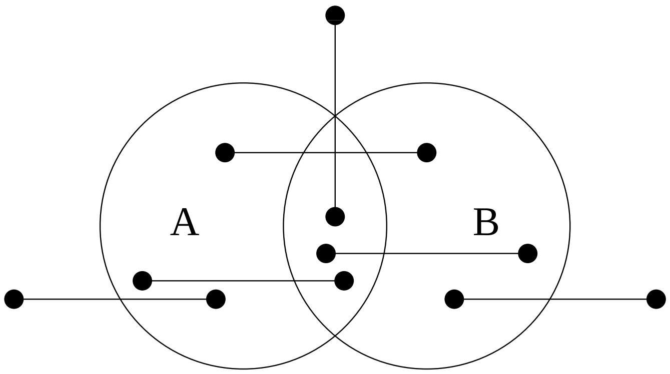

Figure 2: Proof of Lemma 3.3

Corollary 3.4 *Let $G = (V, E)$ be a graph and $X, Y \subseteq V$. If there are p commodities with one terminal pair in each of $X \backslash Y, Y \backslash X$, then*

$$|D(X)| + |D(Y)| = |D(X \cup Y)| + |D(X \cap Y)| + 2p.$$

Proof. Form graph $G' = (V, E')$ as follows: for $v, w \in V$, include edge (v, w) in E if and only if v and w are terminals pairs of a common commodity in G. The corollary follows by applying Lemma 3.3 to G'. ■

Call a vertex set $X \subseteq V$ *tight* if $|\delta(X)| = |D(X)|$ (i.e., sets for which the cut condition "barely holds"). Our final lemma shows that in graphs for which the cut condition holds, tight sets that do not "split" any commodities have a nice nested structure.

Lemma 3.5 *Suppose $G = (V, E)$ is a graph in which the cut condition holds. Suppose $X, Y \subseteq V$ are tight sets and that no commodity has exactly one of its terminal in each of $X \backslash Y, Y \backslash X$. Then $X \cup Y$ and $X \cap Y$ are also tight sets, and no edges of E have one endpoint in each of $X \backslash Y, Y \backslash X$.*

Proof. Let X, Y be tight sets with no commodities having terminals split between $X \backslash Y$ and $Y \backslash X$, and suppose there are $p \geq 0$ edges with one endpoint in each of $X \backslash Y, Y \backslash X$. We then have

$$\begin{aligned} |D(X \cup Y)| + |D(X \cap Y)| &= |D(X)| + |D(Y)| && (1) \\ &= |\delta(X)| + |\delta(Y)| && (2) \\ &= |\delta(X \cup Y)| + |\delta(X \cap Y)| + 2p && (3) \\ &\geq |D(X \cup Y)| + |D(X \cap Y)| + 2p && (4) \end{aligned}$$

where (1) follows from Corollary 3.4 and the assumption that no commodity has one terminal pair in each of $X \backslash Y, Y \backslash X$, (2) follows from the assumption that X, Y are tight, (3) follows from Lemma 3.3, and (4) follows from the assumption that cut condition holds. Since the final inequality must be an equality, we have $|D(X \cup Y)| = |\delta(X \cup Y)|$, $|D(X \cap Y)| = |\delta(X \cap Y)|$, and $p = 0$. ■

We remark that all of the results of this section hold in more general settings; in particular, nowhere have we assumed that our graphs are planar.